Lecture Notes for IEOR 266: Graph Algorithms and Network Flows

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Contents

These notes are based on "scribe" notes taken by students attending Professor Hochbaum's course IEOR 266. The current version has been updated and edited by Professor Hochbaum in 2014.

The text book used for the course, and mentioned in the notes, is *Network Flows: theory, algorithms and applications* by Ravindra K. Ahuja, Thomas L. Magnanti and James B. Orlin. Published by Prentice-Hall, 1993. The notes also make reference to the book *Combinatorial Optimization: algorithms and complexity* by Christos H. Papadimitriou and Kenneth Steiglitz, published by Prentice Hall, 1982.

1 Introduction

We will begin the study of network flow problems with a review of the formulation of linear programming (LP) problems. Let the number of decision variables, x_j's, be N, and the number of constraints be M. LP problems take the following generic form:

$$\begin{array}{lll} \min & \sum_{j=1}^{N} c_j x_j & \\ \text{s.t.} & \sum_{j=1}^{N} a_{i,j} x_j \leq b_i & \forall i \in \{1,\ldots,M\} \\ & x_j \geq 0 & \forall j \in \{1,\ldots,N\} \end{array} \tag{1}$$

Integer linear programming (ILP) has the following additional constraint:

$$x_j \text{ integer} \quad \forall j \in \{1,\ldots,N\} \tag{2}$$

It may appear that ILP problems are simpler than LP problems, since the solution space in ILP is countably finite while the solution space in LP is infinite; one obvious solution to ILP is *enumeration*, i.e. systematically try out all feasible solutions and find one with the minimum cost. As we will see in the following ILP example, enumeration is not an acceptable algorithm as, even for moderate-size problems, its running time would be extremely large.

1.1 Assignment problem

In words, the assignment problem is the following: given n tasks to be completed individually by n people, what is the minimum cost of assigning all n tasks to n people, one task per person, where $c_{i,j}$ is the cost of assigning task j to person i? Let the decision variables be defined as:

$$x_{i,j} = \begin{cases} 1 & \text{if person } i \text{ takes on task } j \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

The problem can now be formulated as:

$$\begin{array}{lll} \min & \sum_{j=1}^{n} \sum_{i=1}^{n} c_{i,j} x_{i,j} & \\ \text{s.t.} & \sum_{j=1}^{n} x_{i,j} = 1 & \forall i = 1,...,n \\ & \sum_{i=1}^{n} x_{i,j} = 1 & \forall j = 1,...,n \\ & x_{i,j} \geq 0 & x_{i,j} \text{ integer} \end{array} \tag{4}$$

The first set of constraints ensures that exactly one person is assigned to each task; the second set of constraints ensures that each person is assigned exactly one task. Notice that the upper bound constraints on the x_{ij} are unnecessary. Also notice that the set of constraints is not independent (the sum of the first set of constraints equals the sum of the second set of constraints), meaning that one of the $2n$ constraint can be eliminated.

While at first it may seem that the integrality condition limits the number of possible solutions and could thereby make the integer problem easier than the continuous problem, the opposite is actually true. Linear programming optimization problems have the property that there exists an optimal solution at a so-called *extreme point* (a basic solution); the optimal solution in an integer program, however, is not guaranteed to satisfy any such property and the number of possible integer valued solutions to consider becomes prohibitively large, in general.

Consider a simple algorithm for solving the Assignment Problem: It enumerates all possible assignments and takes the cheapest one. Notice that there are $n!$ possible assignments. If we

consider an instance of the problem in which there are 70 people and 70 tasks, that means that there are

$$\begin{aligned} 70! &= 11978571669969891796072783721689098736458938142546425857555362864628009582789845319680000000000000000 \\ &\approx 2^{332} \end{aligned}$$

different assignments to consider. Our simple algorithm, while correct, is not at all practical! The existence of an algorithm does not mean that there exits a useful algorithm.

While linear programming belongs to the class of problems P for which "good" algorithms exist (an algorithm is said to be good if its running time is bounded by a polynomial in the size of the input), integer programming belongs to the class of *NP*-hard problems for which it is considered highly unlikely that a "good" algorithm exists. Fortunately, as we will see later in this course, the Assignment Problem belongs to a special class of integer programming problems known as the *Minimum Cost Network Flow Problem*, for which efficient polynomial algorithms *do* exist.

The reason for the tractability of the assignment problem is found in the form of the constraint matrix. The constraint matrix is *totally unimodular* (TUM). Observe the form of the constraint matrix A:

$$A = \begin{bmatrix} 0 & 1 & \cdots \\ 1 & 0 & \cdots \\ 0 & \vdots & \cdots \\ \vdots & \vdots & \\ 1 & \vdots & \\ 0 & 1 & \end{bmatrix} \tag{5}$$

We first notice that each column contains exactly two 1's. Notice also that the rows of matrix A can be partitioned into two sets, say A_1 and A_2, such the two 1's of each column are in different sets. It turns out these two conditions are sufficient for the matrix A to be TUM. Note that simply having 0's and 1's are not sufficient for the matrix to be TUM. Consider the constraint matrix for the *vertex cover* problem:

$$A = \begin{bmatrix} 0 & \cdots & 1 & 0 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 1\,0 & \cdots \\ \vdots & & & & & \end{bmatrix} \tag{6}$$

Although this matrix also contains 0's and 1's only, it is not TUM. In fact, vertex cover is an *NP*-complete problem. We will revisit the vertex cover problem in the future. We now turn our attention to the formulation of the generic minimum cost network flow problem.

1.2 Basic graph definitions

- A graph or undirected graph G is an ordered pair $G := (V, E)$. Where V is a set whose elements are called vertices or nodes, and E is a set of unordered pairs of vertices of the form $[i, j]$, called edges.

- A directed graph or digraph G is an ordered pair $G := (V, A)$. Where V is a set whose elements are called vertices or nodes, and A is a set of ordered pairs of vertices of the form (i, j), called arcs. In an arc (i, j) node i is called the *tail* of the arc and node j the *head* of the arc. We sometimes abuse of the notation and refer to a digraph also as a graph.

- A path (directed path) is an ordered list of vertices $(v_1, \ldots, v_k)$, so that $(v_i, v_{i+1}) \in E$ $((v_i, v_{i+1}) \in A)$ for all $i = 1 \ldots, k$. The length of a path is $|(v_1, \ldots, v_k)| = k$.

- A cycle (directed cycle) is an ordered list of vertices $v_0, \ldots, v_k$, so that $(v_i, v_{i+1}) \in E$ $((v_i, v_{i+1}) \in A)$ for all $i = 1, 2, \ldots, n$ and $v_0 = v_k$. The length of a cycle is $|(v_0, \ldots, v_k)| = k$.

- A simple path (simple cycle) is a path (cycle) where all vertices $v_1, \ldots, v_k$ are distinct.

- An (undirected) graph is said to be *connected* if, for every pair of nodes, there is an (undirected) path starting at one node and ending at the other node.

- A directed graph is said to be *strongly connected* if, for every (ordered) pair of nodes (i, j), there is a directed path in the graph starting in i and ending in j.

- The *degree* of a vertex is the number of edges incident to the vertex. In the homework assignement you will prove that $\sum_{v \in V} degree(v) = 2|E|$.

- In a directed graph the *indegree* of a node is the number of incoming arcs taht have that node as a head. The outdegree of a node is the number of outgoing arcs from a node, that have that node as a tail.

  Be sure you can prove,

  $$\sum_{v \in V} indeg(v) = |A|,$$

  $$\sum_{v \in V} outdeg(v) = |A|.$$

- A *tree* can be characterized as a connected graph with no cycles. The relevant property for this problem is that a tree with n nodes has $n - 1$ edges.

  Definition: An undirected graph $G = (V, T)$ is a tree if the following three properties are satisfied:

  Property 1: $|T| = |V| - 1$.
  Property 2: G is connected.
  Property 3: G is acyclic.

  (Actually, it is possible to show, that any two of the properties imply the third).

- A graph is *bipartite* if the vertices in the graph can be partitioned into two sets in such a way that no edge joins two vertices in the same set.

- A *matching* in a graph is set of graph edges such that no two edges in the set are incident to the same vertex.
  The *bipartite (nonbipartite) matching problem*, is stated as follows: Given a bipartite (nonbipartite) graph $G = (V, E)$, find a maximum cardinality matching.

2 Totally unimodular matrices

An $m \times n$ matrix is said to be *unimodular* if for every submatrix of size $m \times m$ the value of its determinant is $1, 0$, or -1. A matrix, A, is said to be *totally unimodular* if the determinant of every square submatrix is $1, 0$ or -1.

Consider the general linear programming problem with integer coefficients in the constraints, right-hand sides and objective function:

(LP)

$$\begin{aligned} \max \quad & cx \\ \text{subject to} \quad & Ax \leq b \end{aligned}$$

Recall that a basic solution of this problem represents an extreme point on the polytope defined by the constraints (i.e. cannot be expressed as a convex combination of any two other feasible solutions). Also, if there is an optimal solution for LP then there is an optimal, basic solution. If in addition A is totally unimodular we have the following result.

Lemma 2.1. *If the constraint matrix A in LP is totally unimodular, then there exists an integer optimal solution. In fact every basic or extreme point solution is integral.*

Proof. If A is $m \times n$, $m < n$, a basic solution is defined by a nonsingular, $m \times m$ square submatrix of A, $B = (b_{ij})$, called the basis, and a corresponding subset of the decision variables, x_B, called the basic variables, such that

$$x_B = B^{-1}b.$$

By Cramer's rule, we know

$$B^{-1} = \frac{C^T}{det(B)},$$

where $C = (c_{ij})$ is an $m \times m$ matrix, called the cofactor matrix of B. Each c_{ij} is the determinant of a submatrix formed by by deleting row i and column j from B; this determinant is then multiplied by an appropriate sign coefficient (1 or -1) depending on whether $i + j$ is even or odd. That is,

$$C_{ij} = (-1)^{i+j} \cdot det \begin{pmatrix} b_{1,1} & \dots & b_{1,j-1} & X & b_{1,j+1} & \dots & b_{1,m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{i-1,1} & \dots & b_{i-1,j-1} & X & b_{i-1,j+1} & \dots & b_{i-1,m} \\ X & X & X & X & X & X & X \\ b_{i+1,1} & \dots & b_{i+1,j-1} & X & b_{i+1,j+1} & \dots & b_{i+1,m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{m,1} & \dots & b_{m,j-1} & X & b_{m,j+1} & \dots & b_{m,m} \end{pmatrix}.$$

Thus, since the determinant is a sum of terms each of which is a product of entries of the submatrix, then all entries in C are integer. In addition, A is totally unimodular and B is a basis (so $|det(B)| = 1$). Therefore, all of the elements of B^{-1} are integers. So we conclude that any basic solution $x_B = B^{-1}b$ is integral. □

Although we will require all coefficients to be integers, the lemma also holds if only the right hand sides vector b is integer.

This result has ramifications for combinatorial optimization problems. Integer programming problems with totally unimodular constraint matrices can be solved simply by solving the linear

programming relaxation. This can be done in polynomial time via the Ellipsoid method, or more practically, albeit not in polynomial time, by the simplex method. The optimal basic solution is guaranteed to be integral.

The set of totally unimodular integer problems includes network flow problems. The rows (or columns) of a network flow constraint matrix contain exactly one 1 and exactly one -1, with all the remaining entries 0. A square matrix A is said to have the *network property* if every column of A has at most one entry of value 1, at most one entry of value -1, and all other entries 0.

3 Minimum cost network flow problem

We now formulate the Minimum Cost Network Flow Problem. In general,

Problem Instance: Given a network $G = (N, A)$, with a cost c_{ij}, upper bound u_{ij}, and lower bound l_{ij} associated with each directed arc (i, j), and supply b_v at each node. Find the cheapest integer valued *flow* such that it satisfies: 1) the capacity constraints on the arcs, 2) the supplies/demands at the nodes, 3) that the flow is conserved through the network.

Integer Programming Formulation: The problem can be formulated as n IP in the following way:

$$\begin{array}{rll} \min & \sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}x_{ij} & \\ \text{subject to} & \sum_{k\in N} x_{ik} - \sum_{k\in N} x_{ki} = b_i & \forall i \in N \\ & l_{ij} \le x_{ij} \le u_{ij}, & x_{ij} \text{ integer}, \ \forall (i,j) \in A \end{array}$$

The first set of constraints is formed by the flow balance constraints. The second set is formed by the the capacity constraints. Nodes with negative supply are sometimes referred to as *demand* nodes and nodes with 0 supply are sometimes referred to as *transshipment* nodes. Notice that the constraints are dependent (left hand sides add to 0, so that the right hand sides must add to for consistency) and, as before, one of the constraints can be removed.

The distinctive feature of this matrix of the flow balance constraint matrix is that each column has precisely one 1 and one -1 in it. Can you figure out why?

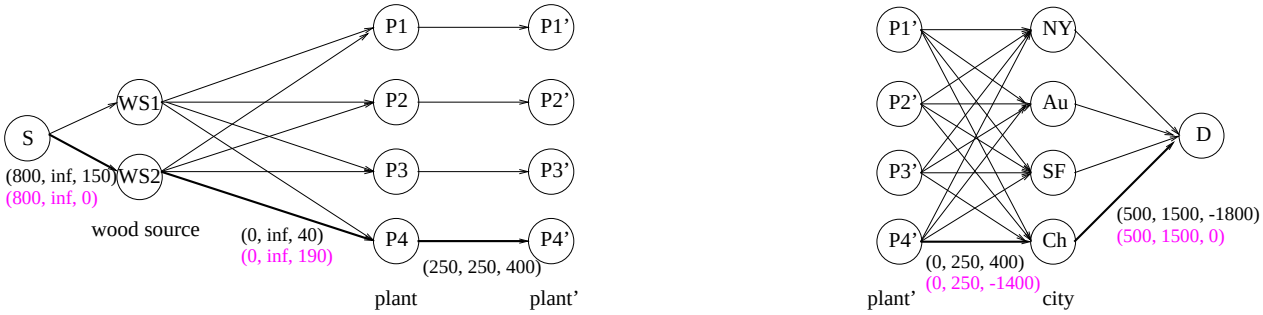
$$A = \begin{bmatrix} 0 & 1 & \cdots \\ 0 & -1 & \cdots \\ 1 & 0 & \\ 0 & \vdots & \\ -1 & \vdots & \\ 0 & & \\ \vdots & & \end{bmatrix} \tag{7}$$

It turns out that all extreme points of the linear programming relaxation of the Minimum Cost Network Flow Problem are integer valued (assuming that the supplies and upper/lower bounds are integer valued). This matrix, that has one 1 and one -1 per column (or row) is *totally unimodular* and therefore MCNF problems can be solved using LP techniques. However, we will investigate more efficient techniques to solve MCNF problems, which found applications in many areas. In fact, the assignment problem is a special case of MCNF problem, where each person is represented by a supply node of supply 1, each job is represented by a demand node of demand 1, and there are

an arc from each person j to each job i, labeled by $c_{i,j}$, that denotes the cost of person j performing job i.

Example of general MCNF problem - the chairs problem

We now look at an example of general network flow problem, as described in handout #2. We begin the formulation of the problem by defining the units of various quantities in the problem: we let a cost unit be 1 cent, and a flow unit be 1 chair. To construct the associated graph for the problem, we first consider the wood sources and the plants, denoted by nodes $WS1, WS2$ and $P1, \ldots, P4$ respectively, in Figure 1a. Because each wood source can supply wood to each plant, there is an arc from each source to each plant. To reflect the transportation cost of wood from wood source to plant, the arc will have cost, along with capacity lower bound (zero) and upper bound (infinity). Arc $(WS2, P4)$ in Figure 1a is labeled in this manner.



(a) wood source & plants

(b) plants & cities

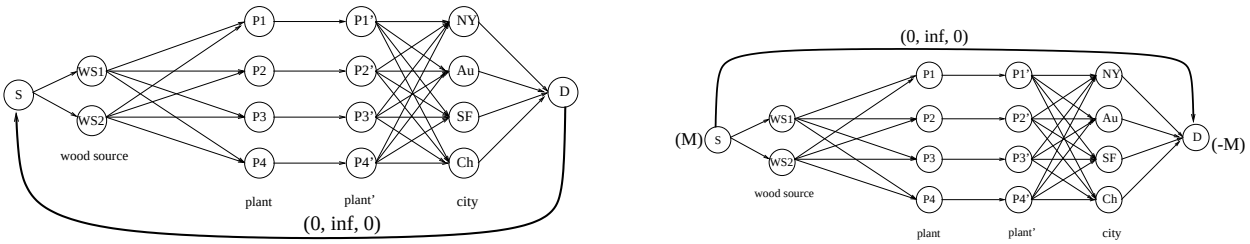
Figure 1: Chair Example of MCNF

It is not clear exactly how much wood each wood source should supply, although we know that contract specifies that each source must supply at least 800 chairs worth of wood (or 8 tons). We represent this specification by *node splitting*: we create an extra node S, with an arc connecting to each wood source with capacity lower bound of 800. Cost of the arc reflects the cost of wood at each wood source. Arc $(S, WS2)$ is labeled in this manner. Alternatively, the cost of wood can be lumped together with the cost of transportation to each plant. This is shown as the grey color labels of the arcs in the figure.

Each plant has a production maximum, production minimum and production cost per unit. Again, using the node splitting technique, we create an additional node for each plant node, and labeled the arc that connect the original plant node and the new plant node accordingly. Arc $(P4, P4')$ is labeled in this manner in the figure.

Each plant can distribute chairs to each city, and so there is an arc connecting each plant to each city, as shown in Figure 1b. The capacity upper bound is the production upper bound of the plant; the cost per unit is the transportation cost from the plant to the city. Similar to the wood sources, we don't know exactly how many chairs each city will demand, although we know there is a demand upper and lower bound. We again represent this specification using the node splitting technique; we create an extra node D and specify the demand upper bound, lower bound and selling price of each unit of each city by labeling the arc that connects the city and node D.

Finally, we need to relate the supply of node S to the demand of node D. There are two ways to accomplish this. We observe that it is a closed system; therefore the number of units supplied at node S is exactly the number of units demanded at node D. So we construct an arc from node



(a) circulation arc

(b) drainage arc

Figure 2: Complete Graph for Chair Example

D to node S, with label $(0, \infty, 0)$, as shown in Figure 2a. The total number of chairs produced will be the number of flow units in this arc. Notice that all the nodes are now transshipment nodes — such formulations of problems are called *circulation problems*.

Alternatively, we can supply node S with a large supply of M units, much larger than what the system requires. We then create a *drainage arc* from S to D with label $(0, \infty, 0)$, as shown in Figure 2b, so the excess amount will flow directly from S to D. If the chair production operation is a money losing business, then all M units will flow directly from S to D.

3.1 Transportation problem

A problem similar to the minimum cost network flow, which is an important special case is called the *transportation problem*: the nodes are partitioned into suppliers and consumers only. There are no transhippment nodes. All edges go directly from suppliers to consumers and have infinite capacity and known per unit cost. The problem is to satisfy all demands at minimum cost. The assignment problem is a special case of the transportation problem, where all supply values and all demand values are 1.

The graph that appears in the transportation problem is a so-called **bipartite** graph, or a graph in which all vertices can be divided into two classes, so that no edge connects two vertices in the same class:

$$\begin{aligned} (V_1 \cup V_2; A) \qquad & \forall (u, v) \in A \\ & u \in V_1, \quad v \in V_2 \end{aligned} \tag{8}$$

A property of bipartite graphs is that they are 2-colorable, that is, it is possible to assign each vertex a "color" out of a 2-element set so that no two vertices of the same color are connected.

An example transportation problem is shown in Figure 3. Let s_i be the supplies of nodes in V_1, and d_j be the demands of nodes in V_2. The transportation problem can be formulated as an IP problem as follows:

$$\begin{array}{lll} \min & \sum_{i \in V_1, j \in V_2} c_{i,j} x_{i,j} & \\ \text{s.t.} & \sum_{j \in V_2} x_{i,j} = s_i & \forall i \in V_1 \\ & -\sum_{i \in V_1} x_{i,j} = -d_j & \forall j \in V_2 \\ & x_{i,j} \geq 0 & x_{i,j} \text{ integer} \end{array} \tag{9}$$

Note that assignment problem is a special case of the transportation problem. This follows from the observation that we can formulate the assignment problem as a transportation problem with $s_i = d_j = 1$ and $|V_1| = |V_2|$.

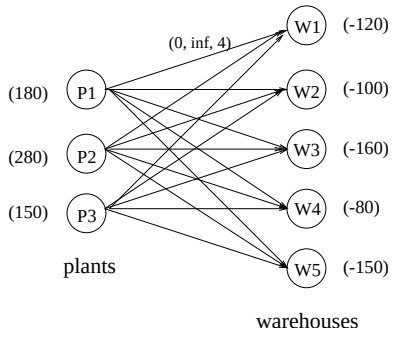


Figure 3: Example of a Transportation Problem

3.2 The maximum flow problem

Recall the IP formulation of the *Minimum Cost Network Flow* problem.

$$\begin{array}{lll} & \text{Min} & cx \\ (\text{ MCNF}) & \text{subject to} & Tx = b \qquad \text{Flow balance constraints} \\ & & l \le x \le u, \qquad \text{Capacity bounds} \end{array}$$

Here $x_{i,j}$ represents the flow along the arc from i to j. We now look at some specialized variants.

The *Maximum Flow* problem is defined on a directed network with capacities u_{ij} on the arcs, and no costs. In addition two nodes are specified, a source node, s, and sink node, t. The objective is to find the maximum flow possible between the source and sink while satisfying the arc capacities. If $x_{i,j}$ represents the flow on arc (i, j), and A the set of arcs in the graph, the problem can be formulated as:

$$\begin{array}{lll} & \text{Max} & V \\ & \text{subject to} & \sum_{(s,i)\in A} x_{s,i} = V \qquad \text{Flow out of source} \\ (\text{MaxFlow}) & & \sum_{(j,t)\in A} x_{j,t} = V \qquad \text{Flow in to sink} \\ & & \sum_{(i,k)\in A} x_{i,k} - \sum_{(k,j)\in A} x_{k,j} = 0, \qquad \forall k \ne s,t \\ & & l_{i,j} \le x_{i,j} \le u_{i,j}, \quad \forall (i,j) \in A. \end{array}$$

The *Maximum Flow* problem and its dual, known as the *Minimum Cut* problem have a number of applications and will be studied at length later in the course.

3.3 The shortest path problem

The *Shortest Path* problem is defined on a directed, weighted graph, where the weights may be thought of as distances. The objective is to find a path from a source node, s, to node a sink node, t, that minimizes the sum of weights along the path. To formulate as a network flow problem, let $x_{i,j}$ be 1 if the arc (i, j) is in the path and 0 otherwise. Place a supply of one unit at s and a demand of one unit at t. The formulation for the problem is:

$$\begin{array}{lll} & \text{Min} & \sum_{((i,j)\in A} d_{i,j} x_{i,j} \\ & \text{subject to} & \sum_{(i,k)\in A} x_{i,k} - \sum_{(j,i)\in A} x_{j,i} = 0 \qquad \forall i \ne s,t \\ (\text{SP}) & & \sum_{(s,i)\in A} x_{s,i} = 1 \\ & & \sum_{(j,t)\in A} x_{j,t} = 1 \\ & & 0 \le x_{i,j} \le 1 \end{array}$$

This problem is often solved using a variety of search and labeling algorithms depending on the structure of the graph.

3.4 The single source shortest paths

A related problem is the *single source shortest paths* problem (SSSP). The objective now is to find the shortest path from node s to all other nodes $t \in V \setminus s$. The formulation is very similar, but now a flow is sent from s, which has supply $n-1$, to every other node, each with demand one.

$$\text{(SSSP)} \qquad \begin{array}{ll} \text{Min} & \sum_{(i,j)\in A} d_{i,j}x_{i,j} \\ \text{subject to} & \sum_{(i,k)\in A} x_{i,k} - \sum_{(j,i)\in A} x_{j,i} = -1 \qquad \forall i \neq s \\ & \sum_{(s,i)\in A} x_{s,i} = n-1 \\ & 0 \leq x_{i,j} \leq n-1, \quad \forall (i,j) \in A. \end{array}$$

But in this formulation, we are minimizing the sum of all the costs to each node. How can we be sure that this is a correct formulation for the single source shortest paths problem?

Theorem 3.1. *For the problem of finding all shortest paths from a single node s, let $\bar{X}$ be a solution and $A^+ = \{(i,j)|X_{ij} > 0\}$. Then there exists an optimal solution such that the in-degree of each node $j \in V - \{s\}$ in $G = (V, A^+)$ is exactly 1.*

Proof. First, note that the in-degree of each node in $V - \{s\}$ is ≥ 1, since every node has at least one path (shortest path to itself) terminating at it.
If every optimal solution has the in-degree of some nodes > 1, then pick an optimal solution that minimizes $\sum_{j\in V-\{s\}} indeg(j)$.
In this solution, i.e. graph $G^+ = (V, A^+)$ take a node with an in-degree > 1, say node i. Backtrack along two of the paths until you get a common node. A common node necessarily exists since both paths start from s.

If one path is longer than the other, it is a contradiction since we take an optimal solution to the LP formulation. If both paths are equal in length, we can construct an alternative solution.

For a path segment, define the bottleneck link as the link in this segment that has the smallest amount of flow. And define this amount as the bottleneck amount for this path segment. Now consider the two path segments that start at a common node and end at i, and compare the bottleneck amounts for them. Pick either of the bottleneck flow. Then, move this amount of flow to the other segment, by subtracting the amount of this flow from the in-flow and out-flow of every node in this segment and adding it to the in-flow and out-flow of every node in the other segment. The cost remains the same since both segments have equal cost, but the in-degree of a node at the bottleneck link is reduced by 1 since there is no longer any flow through it.

We thus have a solution where $\sum_{j\in V-\{s\}} indeg(j)$ is reduced by 1 unit This contradiction completes the proof. □

The same theorem may also be proved following Linear Programming arguments by showing that a basic solution of the algorithm generated by a linear program will always be acyclic in the undirected sense.

3.5 The bipartite matching problem

A matching is defined as M, a subset of the edges E, such that each node is incident to at most one edge in M. Given a graph $G = (V, E)$, find the largest cardinality matching. Maximum cardinality matching is also referred to as the Edge Packing Problem. (We know that $M \leq \lfloor \frac{V}{2} \rfloor$.)

Maximum Weight Matching: We assign weights to the edges and look for maximum total weight instead of maximum cardinality. (Depending on the weights, a maximum weight matching may not be of maximum cardinality.)

Smallest Cost Perfect Matching: Given a graph, find a perfect matching such that the sum of weights on the edges in the matching is minimized. On bipartite graphs, finding the smallest cost perfect matching is the assignment problem.

Bipartite Matching, or *Bipartite Edge-packing* involves pairwise association on a bipartite graph. In a bipartite graph, $G = (V, E)$, the set of vertices V can be partitioned into V_1 and V_2 such that all edges in E are between V_1 and V_2. That is, no two vertices in V_1 have an edge between them, and likewise for V_2. A matching involves choosing a collection of arcs so that no vertex is adjacent to more than one edge. A vertex is said to be "matched" if it is adjacent to an edge. The formulation for the problem is

$$\begin{array}{lll} & \text{Max} & \sum_{(i,j)\in E} x_{i,j} \\ \text{(BM)} & \text{subject to} & \sum_{(i,j)\in E} x_{i,j} \leq 1 \qquad \forall i \in V \\ & & x_{i,j} \in \{0,1\}, \quad \forall\ (i,j) \in E. \end{array}$$

Clearly the maximum possible objective value is $\min\{|V_1|, |V_2|\}$. If $|V_1| = |V_2|$ it may be possible to have a *perfect matching*, that is, it may be possible to match every node.

The *Bipartite Matching* problem can also be converted into a maximum flow problem. Add a source node, s, and a sink node t. Make all edge directed arcs from, say, V_1 to V_2 and add a directed arc of capacity 1 from s to each node in V_1 and from each node in V_2 to t. Place a supply of $\min\{|V_1|, |V_2|\}$ at s and an equivalent demand at t. Now maximize the flow from s to t. The existence of a flow between two vertices corresponds to a matching. Due to the capacity constraints, no vertex in V_1 will have flow to more than one vertex in V_2, and no vertex in V_2 will have a flow from more than one vertex in V_1.

Alternatively, one more arc from t back to s with cost of -1 can be added and the objective can be changed to maximizing circulation. Figure 4 depicts the circulation formulation of bipartite matching problem.

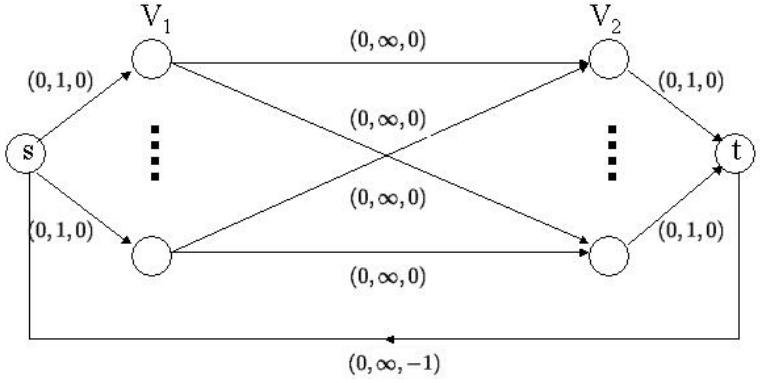


Figure 4: Circulation formulation of Maximum Cardinality Bipartite Matching

The problem may also be formulated as an assignment problem. Figure 5 depicts this formulation. An assignment problem requires that $|V_1| = |V_2|$. In order to meet this requirement we add dummy nodes to the deficient component of B as well as dummy arcs so that every node dummy or original may have its demand of 1 met. We assign costs of 0 to all original arcs and a cost of 1

to all dummy arcs. We then seek the assignment of least cost which will minimize the number of dummy arcs and hence maximize the number of original arcs.

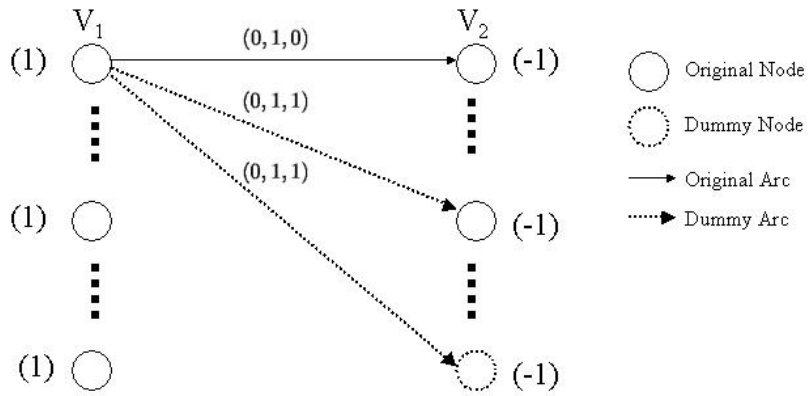


Figure 5: Assignment formulation of Maximum Cardinality Bipartite Matching

The *General Matching*, or *Nonbipartite Matching* problem involves maximizing the number of vertex matching in a general graph, or equivalently maximizing the size of a collection of edges such that no two edges share an endpoint. This problem cannot be converted to a flow problem, but is still solvable in polynomial time.

Another related problem is the weighted matching problem in which each edge $(i, j) \in E$ has weight $w_{i,j}$ assigned to it and the objective is to find a matching of maximum total weight. The mathematical programming formulation of this maximum weighted problem is as follows:

$$\begin{array}{lll} & \text{Max} & \sum_{(i,j)\in E} w_{i,j}x_{i,j} \\ \text{(WM)} & \text{subject to} & \sum_{(i,j)\in E} x_{i,j} \leq 1 \qquad \forall i \in V \\ & & x_{i,j} \in \{0,1\}, \quad \forall\ (i,j) \in E. \end{array}$$

3.6 Summary of classes of network flow problems

There are a number of reasons for classifying special classes of MCNF as we have done in Figure 6. The more specialized the problem, the more efficient are the algorithms for solving it. So, it is important to identify the problem category when selecting a method for solving the problem of interest. In a similar vein, by considering problems with special structure, one may be able to *design* efficient special purpose algorithms to take advantage of the special structure of the problem.

Note that the sets actually overlap, as *Bipartite Matching* is a special case of both the *Assignment Problem* and *Max-Flow*.

3.7 Negative cost cycles in graphs

One of the issues that can arise in weighted graphs is the existence of negative cost cycles. In this case, solving the MCNF problem gives a solution which is unbounded, there is an infinite amount of flow on the subset of the arcs comprising the negative cost cycle.

This problem also occurs if we solve MCNF with upper bounds on the flow on the arcs. To see this, say we modify the formulation with an upper bound on the flow of 1. Then, we can still have a MCNF solution that is not the shortest path, as shown in the Figure 7. In this problem the *shortest path* is $(s, 2, 4, 3, t)$ while the solution to *MCNF* is setting the variables $(x_{2,4}, x_{4,3}, x_{3,2}, x_{s,1}, x_{1,t}) = 1$.

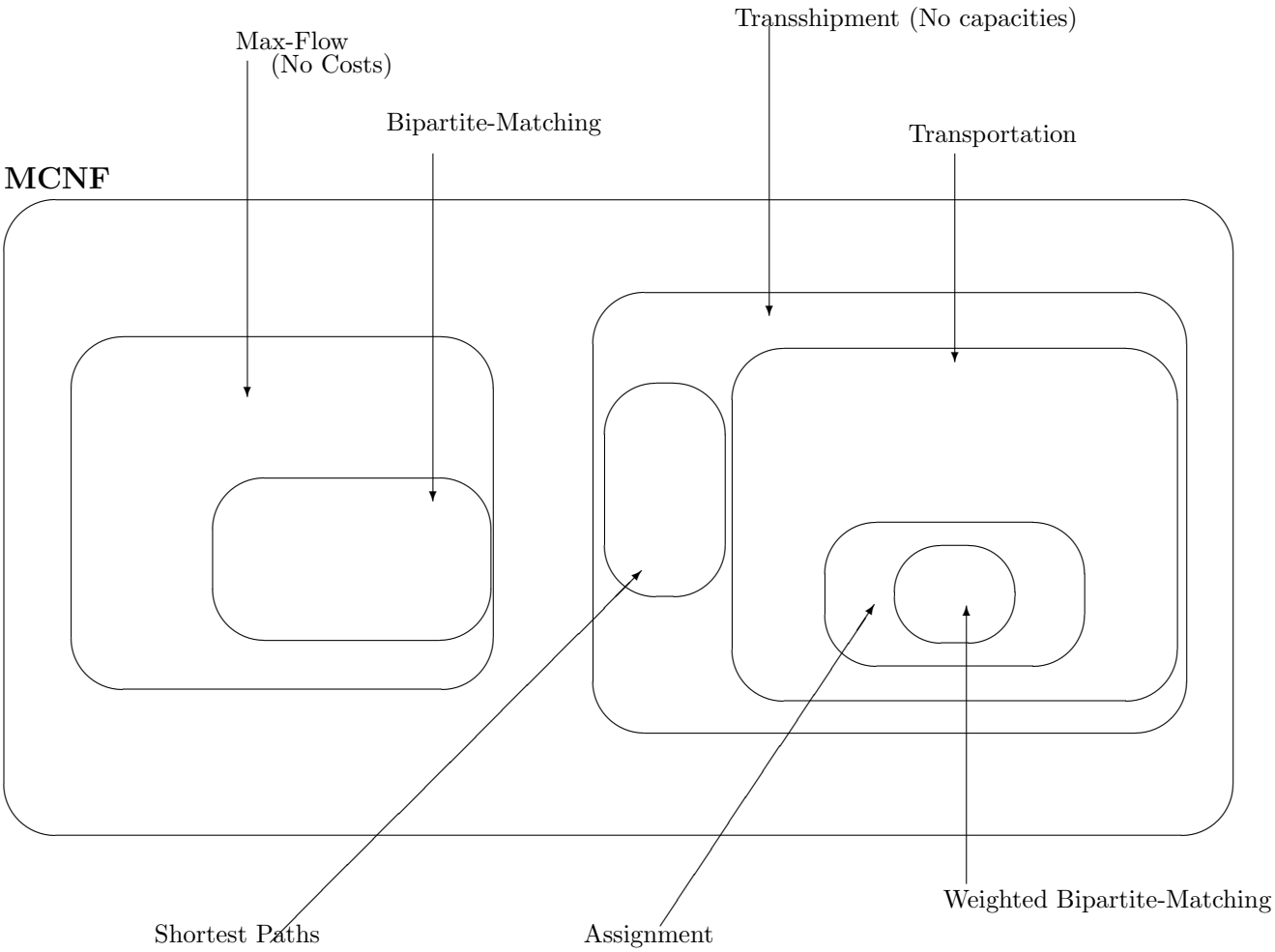


Figure 6: Classification of MCNF problems

These solutions are clearly different, meaning that solving the MCNF problem is not equivalent to solving the shortest path problem for a graph with negative cost cycles.

We will study algorithms for detecting negative cost cycles in a graph later on in the course.

4 Other network problems

4.1 Eulerian tour

We first mention the classcal Königsberg bridge problem. This problem was known at the time of Euler, who solved it in 1736 by posing it as a graph problem. The graph problem is, given an undirected graph, is there a path that traverses all edges exactly once and returns to the starting point. A graph where there exists such path is called an *Eulerian graph.* (You will have an opportunity to show in your homework assignment that a graph is Eulerian if and only if all node degrees are even.)

Arbitrarily select edges forming a trail until the trail closes at the starting vertex. If there are unused edges, go to a vertex that has an unused edge which is on the trail already formed, then create a "detour" - select unused edges forming a new trail until this closes at the vertex you started

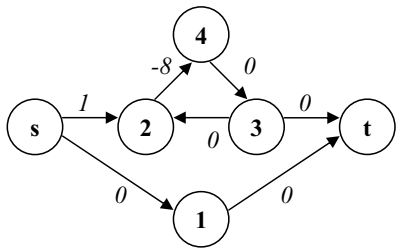


Figure 7: Example of problem with Negative Cost Cycle

with. Expand the original trail by following it to the detour vertex, then follow the detour, then continue with the original trail. If more unused edges are present, repeat this detour construction.

For this algorithm we will use the *adjacency list* representation of a graph, the adjacency lists will be implemented using pointers. That is, for each node i in G, we will have a list of neighbors of i. In these lists each neighbor *points* to the next neighbor in the list and the last neighbor points to a special character to indicated that it is the last neighbor of i:

$$\begin{aligned} n(i) &= (i_1 \rightsquigarrow i_2 \rightsquigarrow \cdots \rightsquigarrow i_k \multimap) \\ d_i &= |\{n(i)\}| \end{aligned}$$

We will maintain a set of positive degree nodes V^+. We also have an ordered list $Tour$, and a *tour pointer* p^T that tells us where to insert the edges that we will be adding to out tour. (In the following pseudocode we will abuse notation and let $n(v)$ refer to the adjacency list of v and also to the first neighbor in the adjacency list of v.)

Pseudocode:

$v \leftarrow 0$ ("initial" node in the tour)
$Tour \leftarrow (0, 0)$
p^T initially indicates that we should insert nodes before the second zero.
While $V^+ \neq \emptyset$ do
    Remove edge $(v, n(v))$ from lists $n(v)$ and $n(n(v))$
    $d_v \leftarrow d_v - 1$
    If $d_v = 0$ then $V^+ \leftarrow V^+ \setminus v$
    Add $n(v)$ to $Tour$, and move p^T to its right
    If $V^+ = \emptyset$ **done**
    If $d_v > 0$ then
        $v \leftarrow n(v)$
    else
        find $u \in Tour \cap V^+$ (with $d_u > 0$)
        set p^T to point to the right of u (that is we must insert nodes right after u)
        $v \leftarrow u$
        end if
end while

4.1.1 Eulerian walk

An Eulerian walk is a path that traverses all edges exactly once (without the restriction of having to end in the starting node). It can be shown that a graph has an Eulerian path if and only if there are exactly 2 nodes of odd degree.

4.2 Chinese postman problem

The Chinese Postman Problem is closely related to the Eulerian walk problem.

4.2.1 Undirected chinese postman problem

Optimization Version: Given a graph $G = (V, E)$ with weighted edges, traverse all edges at least once, so that the total distance traversed is minimized.

This can be solved by first, identifying the odd degree nodes of the graph and constructing a complete graph with these nodes and the shortest path between them as the edges. Then if we can find a minimum weight perfect matching of this new graph, we can decide which edges to traverse twice on the Chinese postman problem and the problem reduces to finding an Eulerian walk.

Decision Version: Given an edge-weighted graph $G = (V, E)$, is there a tour traversing each edge at least once of total weight $\leq M$?

For edge weights = 1, the decision problem with $M = |E| = m$ is precisely the Eulerian Tour problem. It will provide an answer to the question: *Is this graph Eulerian?*

4.2.2 Directed chinese postman problem

Given a gdirected graph $G = (V, A)$ with weighted arcs, traverse all arcs at least once, so that the total distance traversed is minimized.

Note that the necessary and sufficient condition of the existence of a directed Eulerian tour is that the indegree must equal the outdegee on every node. So this can be formulated as a transportation problem as follows.

For each odd degree node, calculate *outdegree-indegree* and this number will be the demand of each node. Then define the arcs by finding the shortest path from each negative demand node to each positive demand node.

4.2.3 Mixed chinese postman problem

Chinese Postman Problem with both directed and undirected edges. Surprisingly, this problem is intractable.

4.3 Hamiltonian tour problem

Given a graph $G = (V, E)$, is there a tour visiting each node exactly once?

This is a *NP*-complete decision problem and no polynomial algorithm is known to determine whether a graph is Hamiltonian.

4.4 Traveling salesman problem (TSP)

Optimization Version: Given a graph $G = (V, E)$ with weights on the edges, visit each node exactly once along a tour, such that we minimize the total traversed weight.

Decision version: Given a graph $G = (V, E)$ with weights on the edges, and a number M, is there a tour traversing each node exactly once of total weight $\leq M$?

4.5 Vertex packing problems (Independent Set)

Given a graph $G = (V, E)$, find a subset S of nodes such that no two nodes in the subset are neighbors, and such that S is of maximum cardinality. (We define neighbor nodes as two nodes with an edge between them.)

We can also assign weights to the vertices and consider the weighted version of the problem.

4.6 Maximum clique problem

A "Clique" is a collection of nodes S such that each node in the set is adjacent to every other node in the set. The problem is to find a clique of maximum cardinality.

Recall that given G, $\bar{G}$ is the graph composed of the same nodes and exactly those edges not in G. Every possible edge is in either G or $\bar{G}$. The maximum independent set problem on a graph G, then, is equivalent to the maximum clique problem in $\bar{G}$.

4.7 Vertex cover problem

Find the smallest collection of vertices S in an undirected graph G such that every edge in the graph is incident to some vertex in S. (Every edge in G has an endpoint in S).

Notation: If V is the set of all vertices in G, and $S \subset V$ is a subset of the vertices, then $V - S$ is the set of vertices in G not in S.

Lemma 4.1. *If S is an independent set, then $V \setminus S$ is a vertex cover.*

Proof. Suppose $V \setminus S$ is not a vertex cover. Then there exists an edge whose two endpoints are not in $V \setminus S$. Then both endpoints must be in S. But, then S cannot be an independent set! (Recall the definition of an independent set – no edge has both endpoints in the set.) So therefore, we proved the lemma by contradiction. □

The proof will work the other way, too:

Result: S is an independent set if and only if $V - S$ is a vertex cover.

This leads to the result that the sum of a vertex cover and its corresponding independent set is a fixed number (namely $|V|$).

Corollary 4.2. *The complement of a maximal independent set is a minimal vertex cover, and vice versa.*

The Vertex Cover problem is exactly the complement of the Independent Set problem.

A 2-Approximation Algorithm for the Vertex Cover Problem: We can use these results to construct an algorithm to find a vertex cover that is at most twice the size of an optimal vertex cover.

(1) Find M, a maximum cardinality matching in the graph.
(2) Let $S = \{u \,|\, u \text{ is an endpoint of some edge in} M\}$.

Claim 4.3. *S is a feasible vertex cover.*

Proof. Suppose S is not a vertex cover. Then there is an edge e which is not covered (neither of its endpoints is an endpoint of an edge in M). So if we add the edge to M ($M \cup e$), we will get a feasible matching of cardinality larger than M. But, this contradicts the maximum cardinality of M. □

Claim 4.4. *$|S|$ is at most twice the cardinality of the minimum VC*

Proof. Let OPT = optimal minimal vertex cover. Suppose $|M| > |OPT|$. Then there exists an edge $e \in M$ which doesn't have an endpoint in the vertex cover. Every vertex in the VC can cover at most one edge of the matching. (If it covered two edges in the matching, then this is not a valid matching, because a match does not allow two edges adjacent to the same node.) This is a contradiction. Therefore, $|M| \leq |OPT|$. Because $|S| = 2|M|$, the claim follows. □

4.8 Edge cover problem

This is analogous to the Vertex cover problem. Given $G = (V, E)$, find a subset $EC \subset E$ such that every node in V is an endpoint of some edge in EC, and EC is of minimum cardinality.

As is often the case, this *edge* problem is of polynomial complexity, even though its *vertex* analog is *NP*-hard.

4.9 b-factor (b-matching) problem

This is a generalization of the Matching Problem: given the graph $G = (V, E)$, find a subset of edges such that each node is incident to at most d edges. Therefore, the Edge Packing Problem corresponds to the 1-factor Problem.

4.10 Graph colorability problem

Given an undirected graph $G = (V, E)$, assign colors $c(i)$ to node $i \in V$, such that $\forall (i, j) \in E;\ c(i) \neq c(j)$.

Example – Map coloring: Given a geographical map, assign colors to countries such that two adjacent countries don't have the same color. A graph corresponding to the map is constructed as follows: each country corresponds to a node, and there is an edge connecting nodes of two neighboring countries. Such a graph is always planar (see definition below).

Definition – Chromatic Number: The smallest number of colors necessary for a coloring is called the *chromatic number* of the graph.

Definition – Planar Graph: Given a graph $G = (V, E)$, if there exists a drawing of the graph in the plane such that no edges cross except at vertices, then the graph is said to be *planar*.

Theorem 4.5 (4-color Theorem)**.** *It is always possible to color a planar graph using 4 colors.*

Proof. A computer proof exists that uses a polynomial-time algorithm (Appel and Haken (1977)). An "elegant" proof is still unavailable. □

Observe that each set of nodes of the same color is an independent set. (It follows that the cardinality of the maximum clique of the graph provides a lower bound to the chromatic number.) A possible graph coloring algorithm consists in finding the maximum independent set and assigning the same color to each node contained. A second independent set can then be found on the subgraph corresponding to the remaining nodes, and so on. The algorithm, however, is both inefficient (finding a maximum independent set is *NP*-hard) and not optimal. It is not optimal for the same reason (and same type of example) that the *b-matching* problem is not solvable optimally by finding a sequence of maximum cardinality matchings.

Note that a 2-colorable graph is bipartite. It is interesting to note that the opposite is also true, i.e., every bipartite graph is 2-colorable. (Indeed, the concept of k-colorability and the property of

being k-partite are equivalent.) We can show it building a 2-coloring of a bipartite graph through the following breadth-first-search based algorithm:

Initialization: Set $i = 1$, $L = \{i\}$, $c(i) = 1$.
General step: Repeat until L is empty:

- For each previously uncolored $j \in N(i)$, (a neighbor of i), assign color $c(j) = c(i) + 1$ $(mod\ 2)$ and add j at the end of the list L. If j is previously colored and $c(j) = c(i) + 1(mod\ 2)$ proceed, else stop and declare *graph is not bipartite.*
- Remove i from L.
- Set $i =$ first node in L.

Termination: Graph is bipartite.

Note that if the algorithm finds any node in its neighborhood already labeled with the opposite color, then the graph is not bipartite, otherwise it is. While the above algorithm solves the 2-color problem in $O(|E|)$, the d-color problem for $d \geq 3$ is *NP*-hard even for planar graphs (see Garey and Johnson, page 87).

4.11 Minimum spanning tree problem

Definition: An undirected graph $G = (V, T)$ is a tree if the following three properties are satisfied:
Property 1: $|T| = |V| - 1$.
Property 2: G is connected.
Property 3: G is acyclic.

(Actually, it is possible to show that any two of the properties imply the third).

Definition: Given an undirected graph $G = (V, E)$ and a subset of the edges $T \subseteq E$ such that (V, T) is tree, then (V, T) is called a *spanning tree* of G.

Minimum spanning tree problem: Given an undirected graph $G = (V, E)$, and costs $c : E \to \Re$, find $T \subseteq E$ such that (V, T) is a tree and $\sum_{(i,j)\in T} c(i, j)$ is minimum. Such a graph (V, T) is a *Minimum Spanning Tree* (MST) of G.

The problem of finding an MST for a graph is solvable by a "greedy" algorithm in $O(|E|log|E|)$. A lower bound on the worst case complexity can be easily proved to be $|E|$, because every arc must get compared at least once. The open question is to see if there exists a linear algorithm to solve the problem (a linear time *randomized* algorithm has been recently devised by Klein and Tarjan, but so far no deterministic algorithm has been provided; still, there are "almost" linear time deterministic algorithms known for solving the MST problem).

5 Complexity analysis

5.1 Measuring quality of an algorithm

Algorithm: One approach is to enumerate the solutions, and select the best one.

Recall that for the assignment problem with 70 people and 70 tasks there are $70! \approx 2^{332.4}$ solutions. The existence of an algorithm does not imply the existence of a good algorithm!

To measure the complexity of a particular algorithm, we count the number of operations that are performed as a function of the 'input size'. The idea is to consider each elementary operation (usually defined as a set of simple arithmetic operations such as $\{+, -, \times, /, <\}$) as having unit cost, and measure the number of operations (in terms of the size of the input) required to solve a

problem. The goal is to measure the rate, ignoring constants, at which the running time grows as the size of the input grows; it is an asymptotic analysis.

Traditionally, complexity analysis has been concerned with counting the number of operations that must be performed in the worst case.

Definition 5.1 (Concrete Complexity of a problem)**.** *The complexity of a problem is the complexity of the algorithm that has the lowest complexity among all algorithms that solve the problem.*

5.1.1 Examples

Set Membership - Unsorted list: We can determine if a particular item is in a list of n items by looking at each member of the list one by one. Thus the number of comparisons needed to find a member in an unsorted list of length n is n.

Problem: given a real number x, we want to know if $x \in S$.

Algorithm:

1. Compare x to s_i
2. Stop if $x = s_i$
3. else if $i \leftarrow i + 1 < n$ goto 1 else stop x is not in S

Complexity = n comparisons in the worst case. This is also the **concrete complexity** of this problem. Why?

Set Membership - Sorted list: We can determine if a particular item is in a list of n elements via **binary search**. The number of comparisons needed to find a member in a sorted list of length n is proportional to $\log_2 n$.

Problem: given a real number x, we want to know if $x \in S$.

Algorithm:

1. Select $s_{med} = \lfloor \frac{first+last}{2} \rfloor$ and compare to x
2. If $s_{med} = x$ stop
3. If $s_{med} < x$ then $S = (s_{med+1}, \ldots, s_{last})$ else $S = (s_{first}, \ldots, s_{med-1})$
4. If $first < last$ goto 1 else stop

Complexity: after k^{th} iteration $\frac{n}{2^{k-1}}$ elements remain. We are done searching for k such that $\frac{n}{2^{k-1}} \leq 2$, which implies:

$$\log_2 n \leq k$$

Thus the total number of comparisons is at most $\log_2 n$.

Aside: This binary search algorithm can be used more generally to find the *zero* in a *monotone increasing* and *monotone nondecreasing* functions.

Matrix Multiplication: The straightforward method for multiplying two $n \times n$ matrices takes n^3 multiplications and $n^2(n-1)$ additions. Algorithms with better complexity (though not necessarily practical, see comments later in these notes) are known. Coppersmith and Winograd (1990) came up with an algorithm with complexity $Cn^{2.375477}$ where C is large. Indeed, the constant term is so large that in their paper Coppersmith and Winograd admit that their algorithm is impractical in practice.

Forest Harvesting: In this problem we have a forest divided into a number of cells. For each cell we have the following information: H_i - benefit for the timber company to harvest, U_i - benefit for the timber company not to harvest, and B_{ij} - the border effect, which is the benefit received for harvesting exactly one of cells i or j. This produces an m by n grid. The way to solve is to look at every possible combination of harvesting and not harvesting and pick the best one. This algorithm requires (2^{mn}) operations.

An algorithm is said to be *polynomial* if its running time is bounded by a polynomial in the size of the input. All but the forest harvesting algorithm mentioned above are polynomial algorithms.

An algorithm is said to be *strongly polynomial* if the running time is bounded by a polynomial in the size of the input *and* is independent of the numbers involved; for example, a max-flow algorithm whose running time depends upon the size of the arc capacities is *not* strongly polynomial, even though it may be polynomial (as in the scaling algorithm of Edmonds and Karp). The algorithms for the sorted and unsorted set membership have strongly polynomial running time. So does the greedy algorithm for solving the minimum spanning tree problem. This issue will be returned to later in the course.

Sorting: We want to sort a list of n items in nondecreasing order.

Input: $S = \{s_1, s_2, \ldots, s_n\}$

Output: $s_{i1} \leq s_{i2} \leq \cdots \leq s_{in}$

Bubble Sort: $n' = n$

While $n' \geq 2$
  $i = 1$
  while $i \leq n' - 1$
    If $s_i > s_{i+1}$ then $t = s_{i+1},\ s_{i+1} = s_i,\ s_i = t$
    $i = i + 1$
  end while
  $n' \leftarrow n' - 1$
end while
Output $\{s_1, s_2, \ldots, s_n\}$

Basically, we iterate through each item in the list and compare it with its neighbor. If the number on the left is greater than the number on the right, we swap the two. Do this for all of the numbers in the array until we reach the end. Then we repeat the process. At the end of the first pass, the last number in the newly ordered list is in the correct location. At the end of the second pass, the last and the penultimate numbers are in the correct positions. And so forth. So we only need to repeat this process a maximum of n times.

The complexity complexity of this algorithm is: $\sum_{k=2}^{n}(k-1) = n(n-1)/2 = O(n^2)$.

Merge sort (a recursive procedure):

Sort(L_n): (this procedure returns L_n^{sort})
begin
  If $n < 2$
    return L_n^{sort}
  else
    partition L_n into $L_{\lfloor n/2 \rfloor}, L_{\lceil n/2 \rceil}$ $(L_n = L_{\lfloor n/2 \rfloor} \cup L_{\lceil n/2 \rceil})$
    Call Sort($L_{\lfloor n/2 \rfloor}$) (to get $L_{\lfloor n/2 \rfloor}^{sort}$)
    Call Sort($L_{\lceil n/2 \rceil}$) (to get $L_{\lceil n/2 \rceil}^{sort}$)
    Call Merge($L_{\lfloor n/2 \rfloor}^{sort}, L_{\lceil n/2 \rceil}^{sort}$)
  end if
end

Merge(L_1,L_2): (this procedure returns L^{sort})
begin
  $L^{sort} \leftarrow empty - list$

```
    While (length(L1)¿0 and length(L2)¿0)
        if first(L1) ≤ first(L2)
            add first(L1) to end of L^sort
            remove first(L1) from L1
        else
            add first(L2) to end of L^sort
            remove first(L2) from L2
        end if
    end while
    if length(L1) > 0
        append L1 to L^sort
    if length(L2) > 0
        append L2 to L^sort
    end if
    return L^sort
end
```

In words, the merge sort algorithm is as follows:

1. If the input sequence has fewer than two elements, return.
2. Partition the input sequence into two halves.
3. Sort the two subsequences using the same algorithm.
4. Merge the two sorted subsequences to form the output sequence.

Analysis:

Size of list	**Comparisons**
2	$n/2$
4	$3 \cdot n/4$
8	$7 \cdot n/8$
...	...
n	$n-1$

We have at most most $\log_2 n$ levels, with at most n comparisons on each. Therefore we perform at most $n \log_2 n$ comparisons.

An alternative way to find the number of comparisons is as follows. The work done in sorting a list of n elements is the work done to sort two lists of size $n/2$ each, and the work done to merge the two lists which is n.

$$f(n) = 2\ f\left(\frac{n}{2}\right) + n$$

Using $f(1) = 1$, we can solve the above recurrence relation, and obtain $f(n) = n \log_2 n$.

We conclude that the complexity complexity of this algorithm is: $n \log_2 n$.

5.2 Growth of functions

n	$\lceil \log n \rceil$	$n-1$	2^n	$n!$
1	0	0	2	1
3	2	2	8	6
5	3	4	32	120
10	4	9	1024	3628800
70	7	69	2^{70}	$\approx 2^{332}$

We are interested in the asymptotic behavior of the running time.

5.3 Definitions for asymptotic comparisons of functions

We define for functions f and g,

$$f, g \ : \ Z^+ \to [0, \infty) \ .$$

1. $f(n) \in O(g(n))$ if $\exists$ a constant $c > 0$ such that $f(n) \leq cg(n)$, for all n sufficiently large.
2. $f(n) \in o(g(n))$ if, for any constant $c > 0$, $f(n) < cg(n)$, for all n sufficiently large.
3. $f(n) \in \Omega(g(n))$ if $\exists$ a constant $c > 0$ such that $f(n) \geq cg(n)$, for all n sufficiently large.
4. $f(n) \in \omega(g(n))$ if, for any constant $c > 0$, $f(n) > cg(n)$, for all n sufficiently large.
5. $f(n) \in \Theta(g(n))$ if $f(n) \in O(g(n))$ and $f(n) \in \Omega(g(n))$.

Examples:

- Bubble sort has complexity $O(n^2)$
- Merge sort has complexity $O(n \log n)$
- Matrix multiplication has complexity $O(n^3)$

5.4 Properties of asymptotic notation

We mention a few properties that can be useful when analyzing the complexity of algorithms.

Proposition 5.2. *$f(n) \in \Omega(g(n))$ if and only if $g(n) \in O(f(n))$.*

The next property is often used in conjunction with *L'hôpital's Rule*.

Proposition 5.3. *Suppose that $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ exists:*

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} \ = \ c \ .$$

Then,

1. *$c < \infty$ implies that $f(n) \in O(g(n))$.*
2. *$c > 0$ implies that $f(n) \in \Omega(g(n))$.*
3. *$c = 0$ implies that $f(n) \in o(g(n))$.*
4. *$0 < c < \infty$ implies that $f(n) \in \Theta(g(n))$.*
5. *$c = \infty$ implies that $f(n) \in \omega(g(n))$.*

An algorithm is *good* or polynomial-time if the complexity is O(polynomial(length of input)). This polynomial must be of fixed degree, that is, its degree must be independent of the input length. So, for example, $O(n^{\log n})$ is not polynomial.

5.5 Caveats of complexity analysis

One should bear in mind a number of caveats concerning the use of complexity analysis.

1. **Ignores the size of the numbers.** The model presented is a poor one when dealing with very large numbers, as each operation is given unit cost, regardless of the size of the numbers involved. But multiplying two huge numbers, for instance, may require more effort than multiplying two small numbers.

2. **Is worst case.** So complexity analysis does not say much about the average case. Traditionally, complexity analysis has been a pessimistic measure, concerned with worst-case behavior. The *simplex method* for linear programming is known to be exponential (in the worst case), while the *ellipsoid* algorithm is polynomial; but, for the ellipsoid method, the average case behavior and the worst case behavior are essentially the same, whereas the average case behavior of simplex is much better than it's worst case complexity, and in practice is preferred to the ellipsoid method.

   Similarly, Quicksort, which has $O(n^2)$ worst case complexity, is often chosen over other sorting algorithms with $O(n \log n)$ worst case complexity. This is because QuickSort has $O(n \log n)$ average case running time and, because the constants (that we ignore in the O notation) are smaller for QuickSort than for many other sorting algorithms, it is often preferred to algorithms with "better" worst-case complexity and "equivalent" average case complexity.

3. **Ignores constants.** We are concerned with the asymptotic behavior of an algorithm. But, because we ignore constants (as mentioned in QuickSort comments above), it may be that an algorithm with better complexity only begins to perform better for instances of inordinately large size.

   Indeed, this is the case for the $O(n^{2.375477})$ algorithm for matrix multiplication, that is *"...wildly impractical for any conceivable applications."* [1]

4. **$\mathbf{O(n^{100})}$ is polynomial.** An algorithm that is *polynomial* is considered to be "good". So an algorithm with $O(n^{100})$ complexity is considered good even though, for reasons already alluded to, it may be completely impractical.

Still, complexity analysis is in general a very useful tool in both determining the intrinsic "hardness" of a *problem* and measuring the quality of a particular *algorithm*.

5.6 A sketch of the ellipsoid method

We now outline the two key features of the ellipsoid method. A linear programming problem may be formulated in one of two ways - a primal and dual representation. The first key feature combines both formulations in a way that forces all feasible solutions to be optimal solutions.

Primal Problem:

$$\max\{c^T x : \ Ax \leq b, \ x \geq 0\} \tag{10}$$

Dual Problem:

$$\min\{b^T y : \ y^T A \geq c, y \geq 0\} \tag{11}$$

The optimal solution satisfies:

$$c^T x = b^T y \tag{12}$$

We create a new problem combining these two formulations:

$$\begin{aligned} Ax &\leq b \\ x &\geq 0 \\ y^T A^T &\geq c \\ y &\geq 0 \\ b^T y &\leq c^T x \end{aligned}$$

[1] See *Coppersmith D. and Winograd S.*, Matrix Multiplication via Arithmetic Progressions. *Journal of Symbolic Computation*, 1990 Mar, V9 N3:251-280.

Note that by weak duality theorem $b^T y \geq c^T x$, so with the inequality above this gives the desired eqaultiy. Every feasible solution to this problem is an optimal solution. The ellipsoid algorithm works on this feasibilty problem as follows:

1. If there is a feasible solution to the problem, then there is a basic feasible solution. Any basic solution is contained in a sphere (ellipsoid) of radius $n2^{\log L}$, where L is the largest subdeterminant of the augmented constraint matrix.
2. Check if the center of the ellipsoid is feasible. If yes, we are done. Otherwise, find a constraint that has been violated.
3. This violated constraint specifies a hyperplane through the center and a half ellipsoid that contains all the feasible solutions.
4. Reduce the search space to the smallest ellipsoid that contains the feasible half ellipsoid.
5. Repeat the algorithm at the center of the new ellipsoid.

Obviously this algorithm is not complete. How do we know when to stop and decide that the feasible set is empty? It's proved that the volume of the ellipsoids scales down in each iteration by a factor of about $e^{-1/2(n+1)}$, where n is the number of linearly independent constraints. Also note that a basic feasible solution only lies on a grid of mesh length $(L(L-1))^{-1}$. Because when we write a component of a basic feasible solution as the quotient of two integers, the denominator cannot exceed L (by Cramer's Rule). Therefore the component-wise difference of two basic feasible solutions can be no less than $(L(L-1))^{-1}$. Given this, it can be shown that the convex hull of basic feasible solutions has volume of at least $2^{-(n+2)L}$. While the volume of the original search space is $2 \times n^2 L^2$. Therefore the algorithm terminates in $O(n^2 \log L)$ iterations. The ellipsoid method is the first polynomial time algorithm for linear programming problems. However, it is not very efficient in practice.

6 Graph representations

There are different ways to store a graph in a computer, and it is important to understand how each representation may affect or improve the complexity analysis of the related algorithms. The differences between each representation are based on the way information is stored, so for instance, one representation may have a characteristic X very handy but characteristic Y may be very operation intensive to obtain, while an alternative representation may favor readiness of characteristic Y over characteristic X.

6.1 Node-arc adjacency matrix

This representation is relevant since it is the natural representation of the flow balance constraints in the LP formulation of Network Flow problems. This representation consists on a n by m matrix, so that each of its rows correspond to a node and each of its columns correspond to an arc. The $(i,j)^{th}$ entry of the Node-arc adjacency matrix is {1,-1}, if i^{th} node is the start-point or endpoint (respectively) of the j^{th} edge and {0} otherwise.

As mentioned in an earlier lecture, the node-arc adjacency matrix is *totally unimodular*.

The density of a node-arc adjacency matrix is $\frac{2}{n}$

6.2 Node-node adjacency matrix

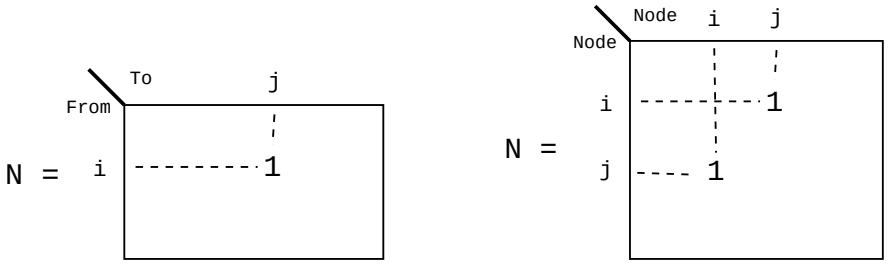
This representation consists on a n by n matrix, so that each of its rows and each of its columns correspond to a node and the row-column intersections correspond to a (possible) edge. The $(i,j)^{th}$

entry of the Node-Node Adjacency Matrix is 1, if there is an edge from the i^{th} node to the j^{th} node and 0 otherwise. In the case of undirected graphs it is easy to see that the Node-Node Adjacency Matrix is a symmetric matrix.

The network is described by the matrix N, whose (i, j)th entry is n_{ij}, where

$$n_{ij} = \begin{cases} 1 & \text{if (i, j)} \in \text{E} \\ 0, & \text{otherwise} \end{cases}$$

Figure 8 depicts the node-node adjacency matrices for directed and undirected graphs.



(a) node-node adjacency matrix for a directed graph

(b) node-node adjacency matrix for an undirected graph

Figure 8: Node-node adjacency matrices

Note: For undirected graphs, the node-node adjacency matrix is symmetric. The density of this matrix is $\frac{2|E|}{|V|^2}$ for an undirected graph, and $\frac{|A|}{|V|^2}$, for a directed graph. If the graph is complete, then $|A| = |V| \cdot |V - 1|$ and the matrix N is dense.

Is this matrix a good representation for a breadth-first-search? (discussed in the next section). To determine whether a node is adjacent to any other nodes, n entries need to be checked for each row. For a connected graph with n nodes and m edges, $n - 1 \leq m \leq n(n - 1)/2$, and there are $O(n^2)$ searches.

6.3 Node-arc adjacency list

Consider a *tree,* which is a very sparse graph with $|E| = |V| - 1$. If we represent an undirected tree using the matrices described above, we use a lot of memory. A more compact representation is to write each node and its associated adjacency list.

This representation consists on n linked lists, each one corresponding to a node (i) and storing all the nodes (j) for which the edge (i,j) is present in the graph. The importance of this representation is that it is the most space efficient and it normally leads to the most efficient algorithms as well.

Node	Adjacent Nodes
1	*Neighbors of node* 1
2	*Neighbors of node* 2
⋮	⋮
n	*Neighbors of node* n

Such a representation requires $2\,|E| + |V|$ memory spaces as opposed to $|E| \cdot |V|$ memory spaces for the matrix representation.

There are 2 entries for every edge or arc, this results in $O(m+n)$ size of the adjacency list representation. This is the most compact representation among the three presented here. For sparse graphs it is the preferable one.

6.4 Comparison of the graph representations

6.4.1 Storage efficiency comparison

Representation	**Storage**	**Non-Zeros**	**Density**	**Notes**
Node-Arc Matrix	$O(mn)$	$O(2m)$	$2/n$	Inefficient
Node-Node Matrix	$O(n^2)$	$O(m)$	m/n^2	Efficient for Dense Graphs
Node-Arc List	$O(m+n)$	0	1	Efficient for Sparse Graphs

6.4.2 Advantages and disadvantages comparison

Representation	**Advantages**	**Disadvantages**
Node-Arc Matrix	Nice Theoretical Properties	Not efficient in practice
Node-Node Matrix	Existence of edge in $O(1)$ time	List neighbors in $O(n)$ time
Node-Arc List	List neighbors in $O(deg(i))$ time	Existence of edge in $O(deg(i))$ time

7 Graph search algorithms

Given a graph $G = (V, A)$ and a source node $s \in V$, we want to check if all the nodes $\in V$ can be reached by s using only the arcs $\in A$. Essentially we want the list of all nodes reachable from s and possibly also the path from s to each node. As seen on class, this set of paths must form a tree, this follows from the correctness-proof of the LP formulation of the single source shortest path problem.

7.1 Generic search algorithm

A generic search algorithm starts with the special node, designates it as the root of a tree, and places it in a list L. The algorithm then selects at each iteration a node from the list, and find a neighbor (out-neighbor for a directed graph), if exists, and adds it to the list. The process continues until all nodes have been visited, or all edges (arcs) have been scanned. The selection mechanism from the list differentiates between search algorithms.

In the procedure below $N_E(v)$ is a neighbor of node v in the graph induced by the set of edges E. For a directed graph, the same algorithm applies with $N_A^+(v)$ referring to an out-neighbor of v with respect to the set of arcs A.

Pseudocode for a generic search algorithm

$L \leftarrow \{root\}$, $V \leftarrow V \setminus \{root\}$;
While $L \neq \emptyset$
  Select $v \in L$ (use selection rule);
  If there exists $N_E(v) = u$, $u \in V \setminus L$,
    $L \leftarrow L \cup \{u\}$; $V \leftarrow V \setminus \{u\}$; $E \leftarrow E \setminus \{[v, u]\}$;
  else $L \leftarrow L \setminus \{v\}$;
  end if
end while

Two basic methods that are used to define the selection mechanism are *Breadth-First-Search* (BFS) and *Depth-First-Search* (DFS).

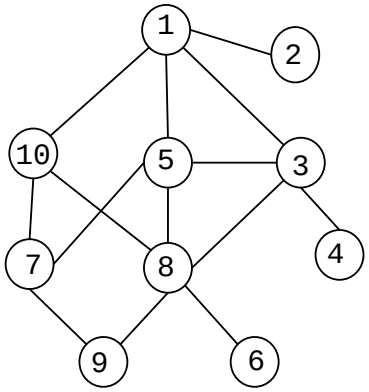


Figure 9: Graph to be searched

7.2 Breadth first search (BFS)

Breadth-first-search employs a *first-in-first-out* (FIFO) strategy and is usually implemented using a queue. To obtain the pseudocode of BFS we only need to modify the pseudocode for the generic search so that when we remove the edge (u, v) from L we remove it from the **start** of L.

In breadth-first-search, the graph is searched in an exhaustive manner; that is, all previously unscanned neighbors of a particular node are tagged and added to the end of the reachability list (L), and then the previously unscanned neighbors of the next node in the list are added, etc.

The original graph, rearranged in a breadth-first-search, shows the typically short and bushy shape of a BFS tree as shown in Figure 10.

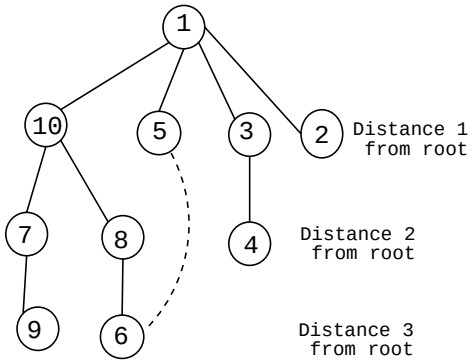


Figure 10: Solid edges represent the BFS tree of the example graph. The dotted edges cannot be present in a BFS tree.

Note that the BFS tree is not unique and depends on the order in which the neighbors are visited. Not all edges from the original graph are represented in the tree. However, we know that the original graph cannot contain any edges that would bridge two nonconsecutive levels in the tree. By level we mean a set of nodes such that the number of edges that connect each node to the root of the tree is the same.

Consider any graph. Form its BFS tree. Any edge belonging to the original graph and not in the BFS can either be from level i to level $i + 1$; or between two nodes at the same level. For

example, consider the dotted edge in Figure 10 from node 5 to node 6. If such an edge existed in the graph, then node 6 would be a child of node 5 and hence would appear at a distance of 2 from the root.

Note that an edge forms an even cycle if it is from level i to level $i+1$ in the graph and an odd cycle if it is between two nodes at the same level.

Theorem 7.1 (BFS Property 1)**.** *The level of a node, $d(v)$, is the distance from s to v.*

Proof. First, it is clear that every element reachable from s will be explored and given a d.

By induction on $d(v)$, Q remains ordered by d, and contains only elements with d equal to k and $k+1$. Indeed, elements with $d = k$ are dequeued first and add elements with $d = k+1$ to Q. Only when all elements with $d = k$ have been used do elements with $d = k+2$ start being introduced. Let "stage k" be the period where elements with $d = k$ are dequeued.

Then, again by induction, at the beginning of stage k, all nodes at distance from s less than k have been enqueued, and only those, and they contain their correct value. Assume this for k. Let v be a node at distance $k+1$. It cannot have been enqueued yet at the beginning of stage k, by assumption, so at that point $d(v) = \infty$. But there exists at least one node u at distance k that leads to u. u is in Q, and $d(u) = k$ by assumption. The smallest such u in Q will give v its value $d(v) = d(u) + 1 = k+1$. Nodes at distance more than $k+1$ have no node at distance k leading to them and will not be enqueued. □

Corollary 7.2 (BFS Property 2)**.** *For undirected graphs $\nexists\{i,j\} : |d(i) - d(j)| > 1$*
(i.e. neighbors must be in adjacent levels)
For directed graphs $\nexists(i,j) : d(j) > d(i) + 1$
(i.e. Can't skip levels in forward arcs)

Claim 7.3 (BFS Complexity)**.** *BFS runs in $O(m+n)$ time.*

7.3 Depth first search (DFS)

Depth-first-search employs a *last-in-first-out* (LIFO) strategy and is usually implemented using a stack. To obtain the pseudocode of BFS we only need to modify the pseudocode for the generic search so that when we remove the edge (u,v) from L we remove it from the **end** of L.

In depth-first-search, one searches for a previously unscanned neighbor of a given node. If such a node is found, it is added to the end of the list and is scanned out from; otherwise, the original node is removed from the list, and we scan from the node now at the end of the list.

The same graph, rearranged in a depth-first-search demonstrates the typically long and skinny shape of a DFS tree as shown in Figure 11.

Claim 7.4 (DFS Complexity)**.** *DFS runs in $O(m+n)$ time.*

7.4 Applications of BFS and DFS

7.4.1 Checking if a graph is strongly connected

Definition 7.5. *A graph is **strongly connected** if we can reach all nodes $\in V$ starting from any other node $\in V$.*

Naive Algorithm

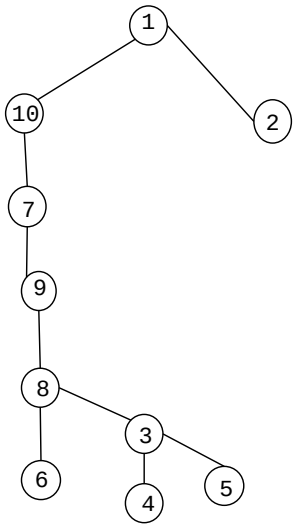


Figure 11: DFS tree of the example graph

```
00: SC(G)
01: For every node in V do
02:    R <-- DFS(G,v); //or BFS
03:    If R not equal to V then STOP AND OUTPUT "FALSE"
04: OUTPUT "TRUE"
```

Since DFS/BFS take $O(m+n)$ time and we have $O(n)$ executions, the complexity of the naive algorithm is $O(mn)$.

Claim 7.6. *Checking strong connectivity in any graph takes $O(m+n)$ time.*

Proof.

For Undirected Graphs

Only one run of DFS or BFS is needed; since, if from any arbitrary node v we can reach all other nodes $\in V$, then we can reach any node from any other node at least through a path containing node v.

For Directed Graphs

From an arbitrarily node v run one DFS or BFS, if it can reach all the other nodes $\in V$, then "flip/reverse" all the edges in G and run once more BFS/DFS starting again from v. It the second time again v can reach all the other nodes $\in V$, then this means that in the original graph every node can reach v. Finally, if every node can reach v, and v can reach every other node, then every node can reach any other node through a path containing v.
In any case we will call DFS/BFS once or twice respectively and this will take $O(m+n)$ time. □

7.4.2 Checking if a graph is acyclic

Definition 7.7. *A graph is **acyclic** if it does not contain cycles.*

Claim 7.8. *Checking if a graph is acyclic takes $O(m+n)$ time.*

Proof. Run DFS from an arbitrarily node, if there is an arc to a node already visited, then the graph contains a cycle. This takes $O(m+n)$ time. □

Although we can use also BFS to detect cycles, DFS tends to find a cycle 'faster'. This is because if the root is not part of the cycle BFS still needs to visit all of its neighbors; while DFS will move away from the root (and hopefully towards the cycle) faster. Also, if the cycle has length k then BFS needs at least to explore floor$(k/2)$ levels.

7.4.3 Checking of a graph is bipartite

Definition 7.9. *A graph is* ***bipartite*** *if V can be partitioned into two sets such that all edges/arcs in E/A have an endpoint on each of the two sets.*

Claim 7.10. *Checking if a graph is bipartite takes $O(m+n)$ time.*

Proof. We know that a graph is bipartite if and only if it does not contains a an odd cycle. So we can run BFS from an arbitrary node v, and if it contains an edge between nodes in the same level then the graph has an odd cycle, and thus it is not bipartite. □

8 Shortest paths

8.1 Introduction

Consider graph $G = (V, A)$, with cost c_{ij} on arc (i, j). There are several different, but related, shortest path problems:

- *Shortest $s-t$ path:* Find a shortest path from s to t (single pair).
- *Single source shortest paths:* Find shortest path from s to all nodes.
- *All pairs shortest paths:* Find the SP between every pair of nodes.

Additionally, we can also differentiate between shortest paths problems depending on the type of graph we receive as input.

We will begin by considering shortest paths (or longest paths) in *Directed Acyclic Graphs* (DAGs).

8.2 Properties of DAGs

A DAG is a Directed Acyclic graph, namely, there are no directed cycles in the graph.

Definition 8.1. *A* Topological ordering *is the assignments of numbering to the nodes $1, 2, \cdots, n$ such that all arcs go in the forward direction:*

$$(i, j) \in E \Rightarrow i < j$$

It should be noted that this graphs can have $O(n^2)$ arcs. We can construct such graph by the following construction: Given a set of nodes, number them arbitrarily and add all arcs (i, j) where $i < j$.

8.2.1 Topological sort and directed acyclic graphs

Given a directed graph (also called a *digraph*), $G = (N, A)$, we wish to *topologically sort* the nodes of G; that is, we wish to assign labels $1, 2, \ldots, |N|$ to the nodes so that $(i, j) \in \mathrm{A} \Rightarrow i < j$.

A topological sort does not exist in general, and a trivial example to show this is a graph with two nodes, with directed edges from each node to the other. It does, however, always exist for acyclic graphs. To prove this, the following definition and lemma will be useful.

Definition 8.2. *A (graph) property, P, is* **hereditary** *if the following is true: given a graph G with property P, any subgraph of G also has property P.*

Some examples of hereditary properties:
- Bipartiteness.
- Having not directed cycles (i.e. being a DAG)

Examples of not hereditary properties:
- Connectivity.
- Being an Eulerian graph.

Lemma 8.3. *In DAG, G, there always exists a node with in-degree* 0*. (Similarly, a DAG always has a node with out-degree* 0*.)*

Proof. Assume by contradiction that G there is no node with in-degree 0. We arbitrarily pick a node i in G, and apply a Depth-First-Search by visiting the predecessor nodes of each such node encountered. Since every node has a predecessor, when the nodes are exhausted (which must happen since there are a finite number of nodes in the graph), we will encounter a predecessor which has already been visited. This shows that there is a cycle in G, which contradicts our assumption that G is acyclic. □

Theorem 8.4. *There exists a topological sort for a digraph G if and only if G is a DAG (i.e. G has no directed cycles).*

Proof.
[Only if part] Suppose there exists a topological sort for G. If there exists cycle (i_1,...,i_k,i_1), then, by the definition of topological sort above, $i_1 < i_2 < ... < i_k < i_1$, which is impossible. Hence, if there exists a topological sort, then G has no directed cycles.
[If part] In a DAG, the nodes can be topologically sorted by using the following procedure. Assigning a node with in-degree 0 the lowest remaining label (by the previous lemma, there always exists such a node). Removing that node from the graph (recall that being a DAG is hereditary), and repeat until all nodes are labeled. The correctness of this algorithm is established by induction on the label number. The pseudocode for the algorithm to find a topological sort is given below. □

Topological sort (pseudocode):
$i \leftarrow 1$
While $V \neq \emptyset$
    Select $v \in V$ with in-degree$_V(v) = 0$
    label$(v) \leftarrow i$
    $i \leftarrow i + 1$
    $V \leftarrow V \setminus \{v\}$; $A \leftarrow A \setminus \{(v, w) \in A\}$
    update in-degrees of neighbors of v
end while

Note that the topological sort resulting from this algorithm is not unique. Further note that we could have alternately taken the node with out-degree 0, assigned it the highest remaining label, removed it from the graph, and iterated on this until no nodes remain.

An example of the topological order obtained for a DAG is depicted in Figure 12. The numbers in the boxes near the nodes represent a topological order.

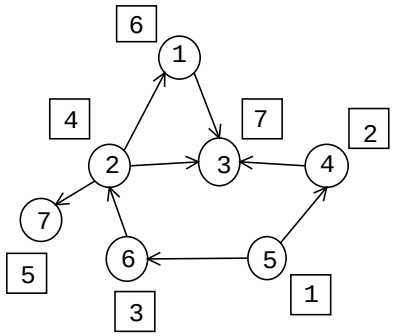


Figure 12: An example of topological ordering in a DAG. Topological orders are given in boxes.

Consider the complexity analysis of the algorithm. We can have a very loose analysis and say that each node is visited at most once and then its list of neighbors has to be updated (a search of $O(m)$ in the node-arc adjacency matrix, or $O(n)$ in the node-node adjacency matrix, or the number of neighbors in the adjacency list data structure). Then need to find among all nodes one with indegree equal to 0. This algorithm runs in $O(mn)$ time, or if we are more careful it can be done in $O(n^2)$. The second factor of n is due to the search for a indegree 0 node. To improve on this we maintain a "bucket" or list of nodes of indegree 0. Each time a node's indegree is updated, we check if it is 0 and if so add it to the bucket. Now to find a node of indegree 0 is done in $O(1)$, just lookup the bucket in any order. Updating the indegrees is done in $O(m)$ time total throughout the algorithm, as each arc is looked at once. Therefore, *Topological Sort* can be implemented to run in $O(m)$ time.

8.3 Properties of shortest paths

The Shortest Path Problem (henceforth SP problem) on a digraph $G = (V, A)$ involves finding the least cost path between two given nodes $s, t \in V$. Each arc $(i, j) \in A$ has a weight assigned to it, and the cost of a path is defined as the sum of the weights of all arcs in the path.

Proposition 8.5. *If the path $(s = i_1, i_2, \ldots, i_h = t)$ is a shortest path from node s to t, then the subpath $(s = i_1, i_2, \ldots, i_k)$ is a shortest path from source to node i_k, $\forall\ k = 2, 3, \ldots, h-1$. Furthermore, for all $k_1 < k_2$ the subpath $(i_{k_1}, i_{k_1+1}, \ldots, i_{k_2})$ is a shortest path from node i_{k_1} to i_{k_2}.*

Proof. The properties given above can be proven by a contradiction-type argument. □

This property is a rephrasing of the principle of optimality (first stated by Richard Bellman in 1952), that serves to justify dynamic programming algorithms.

Proposition 8.6. *Let the vector $\vec{d}$ represent the shortest path distances from the source node. Then*

1. *$d(j) \leq d(i) + c_{ij},\ \forall\,(i, j) \in A$.*
2. *A directed path from s to k is a shortest path if and only if $d(j) = d(i) + c_{ij},\ \forall\,(i, j) \in P$.*

The property given above is useful for backtracking and identifying the tree of shortest paths.

Given the distance vector $\vec{d}$, we call an arc *eligible* if $d(j) = d(i) + c_{ij}$. We may find a (shortest) path from the source s to any other node by performing a breadth first search of eligible arcs. The graph of eligible arcs looks like a tree, plus some additional arcs in the case that the shortest paths are not unique. But we can choose one shortest path for each node so that the resulting graph of all shortest paths from s forms a tree.

Aside: When solving the shortest path problem by linear programming, a basic solution is a tree. This is because a basic solution is an in independent set of columns; and a cycle is a dependent set of columns (therefore "a basic solution cannot contain cycles").

Given and undirected graph with nonnegative weights. The following analogue algorithm solves the shortest paths problem:

The String Solution: Given a shortest path problem on an undirected graph with non-negative costs, imagine physically representing the vertices by small rings (labeled 1 to n), tied to other rings with string of length equaling the cost of each edge connecting the corresponding rings. That is, for each arc (i, j), connect ring i to ring j with string of length c_{ij}.

Algorithm: To find the shortest path from vertex s to all other vertices, hold ring s and let the other nodes fall under the force of gravity.

Then, the distance, $d(k)$, that ring k falls represents the length of the shortest path from s to k. The strings that are taut correspond to eligible arcs, so this arcs will form the tree of shortest paths.

8.4 Alternative formulation for SP from s to t

We have already seen how to model the single source shortest path problem as a Minimum Cost Network Flow problem. Here we will give an alternative mathematical programming formulation for the SP problem.

$$\begin{array}{lcl} \min & & d(t) \\ d(j) & \leq & d(i) + c_{ij} \ \forall (i,j) \in A \\ d(s) & = & 0 \end{array}$$

Where the last constraint is needed as an anchor, since otherwise we would have an infinite number of solutions. To see this, observe that we can rewrite the first constraint as $d(j) - d(i) \leq c_{ij}$; thus, given any feasible solution $\mathbf{d}$, we can obtain an infinite number of feasible solutions (all with same objective value) by adding a constant to all entries of $\mathbf{d}$.

This alternative mathematical programming formulation is nothing else but the dual of the minimum-cost-network-flow-like mathematical programming formulation of the shortest path problem.

8.5 Shortest paths on a directed acyclic graph (DAG)

We consider the single source shortest paths problem for a directed acyclic graph G.

We address the problem of finding the SP between nodes i and j, such that $i < j$, assuming the graph is topologically ordered to start with. It is beneficial to sort the graph in advance, since

we can trivially determine that there is no path between i and j if $i > j$; in other words, we know that the only way to each node j is by using nodes with label less than j. Therefore, without loss of generality, we can assume that the source is labeled 1, since any node with a smaller label than the source is unreachable from the source and can thereby be removed.

Given a topological sorting of the nodes, with node 1 the source, the shortest path distances can be found by using the recurrence,

$$d(j) \;=\; \min_{\substack{(i,j)\in A \\ 1\le i<j}} \{d(i) + c_{ij}\}\,.$$

The validity of the recurrence is easily established by induction on j. It also follows from property 8.6. The above recursive formula is referred to as a dynamic programming equation (or Bellman equation).

What is the complexity of this dynamic programming algorithm? It takes $O(m)$ operations to perform the topological sort, and $O(m)$ operations to calculate the distances via the given recurrence (we obviously compute the recurrence equation in increasing number of node labels). Therefore, the distances can be calculated in $O(m)$ time. Note that by keeping track of the i's that minimize the right hand side of the recurrence, we can easily backtrack to reconstruct the shortest paths.

It is important how we represent the **output** to the problem. If we want a list of all shortest paths, then just writing this output takes $O(n^2)$ (if we list for every node its shortest path). We can do better if we instead just output the shortest path tree (list for every node its predecessor). This way the output length is $O(m)$.

Longest Paths in DAGs:
If we want instead to find the longest path in a DAG, then one possible approach would be to multiply by -1 all the arc costs and use our shortest path algorithm. However, recall that solving the shortest path problem in graphs with negative cost cycles is "a very hard problem". Nevertheless, if the graph where we want to find the longest path is a DAG, then it does not have cycles. Therefore by multiplying by -1 the arc costs we cannot create negative cycles. In conclusion, our strategy will indeed work to find the longest path in a DAG. Alternatively we can find the longest path from node 1 to every other node in the graph, simply by replacing *min* with *max* in the recurrence equations give above.

8.6 Applications of the shortest/longest path problem on a DAG

Maximizing Rent

See the handout titled *Maximizing Rent: Example.*

The graph shown in Figure 13 has arcs going from the arrival day to the departure day with the bid written on the arcs. It is particularly important to have arcs with cost 0 between consecutive days (otherwise we would not allow the place to be empty for some days–which is NOT our objective, what we want is to maximize the rent obtained!). These arcs are shown as dotted arcs in the graph. To find the maximum rent plan one should find the longest path in this graph. The graph has a topological order and can be solved using the shortest path algorithm. (Note that the shortest and longest path problems are equivalent for acyclic networks, but dramatically different when there are cycles.)

Equipment Replacement Minimize the total cost of buying, selling, and operating the equipment over a planning horizon of T periods.

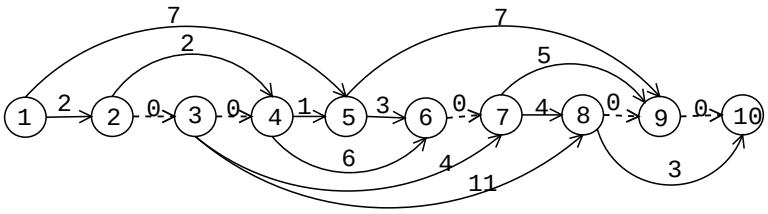


Figure 13: Maximizing rent example

See the handout titled *The Equipment Replacement Problem.*

Lot Sizing Problem Meet the prescribed demand d_j for each of the periods $j = 1, 2, ..., T$ by producing, or carrying inventory from previous production runs. The cost of production includes fixed set up cost for producing a positive amount at any period, and a variable per unit production cost at each period. This problem can be formulated as a shortest path problem in a DAG (AM&O page 749) if one observes that any production will be for the demand of an integer number of consecutive periods ahead. You are asked to prove this in your homework assignment.

Project Management - Critical Path Method (CPM)

See the handout titled *Project Formulation for Designer Gene's Relocation.*

This is a typical project management problem. The objective of the problem is to find the earliest time by which the project can be completed.

This problem can be formulated as a longest path problem. The reason we formulate the problem as a *longest* path problem, and not as a *shortest* path problem (even though we want to find the *earliest* finish time) is that *all* paths in the network have to be traversed in order for the project to be completed. So the longest one will be the "bottleneck" determining the completion time of the project.

We formulate the problem as a longest path problem as follows. We create a node for each task to be performed. We have an arc from node i to node j if activity i must precede activity j. We set the length of arc (i, j) equal to the duration of activity i. Finally we add two spacial nodes *start* and $finish$ to represent the *start* and the *finish* activities (these activities with duration 0). Note that in this graph we cannot have cycles, thus we can solve the problem as a longest path problem in an acyclic network.

Now we give some notation that we will use when solving this type of problems. Each node is labeled as a triplet (node name, time in node, earliest start time). Earliest start time of a node is the earliest time that work can be started on that node. Hence, the objective can also be written as the earliest start time of the finish node.

Then we can solve the problem with our dynamic programming algorithm. That is, we traverse the graph in topological order and assign the label of each node according to our dynamic programming equation:

$$t_j = \max_{(i,j)}\{t_i + c_{ij})\},$$

where t_i is the earliest start time of activity i; and c_{ij} is the duration of activity i.

To recover the longest path, we start from node t_{finish}, and work backwards keeping track of the preceding activity i such that $t_j = t_i + c_{ij}$. Notice that in general more than one activity might satisfy such equation, and thus we may have several longest paths.

Alternatively, we can formulate the longest path as follows:

$$\begin{array}{rcl} \min & & t_{finish} \\ t_j & \geq & t_i + c_{ij} \ \forall (i,j) \in A \\ t_{start} & = & 0 \end{array}$$

From the solution of the linear program above we can identify the longest path as follows. The constraints corresponding to arcs on the longest path will be satisfied with equality. Furthermore, for each unit increase in the length of these arcs our objective value will increase also one unit. Therefore, the dual variables of these constraints will be equal to one (while all other dual variables will be equal to zero–by complementary slackness).

The longest path (also known as critical path) is shown in Figure 14 with thick lines. This path is called critical because any delay in a task along this path delays the whole project path.

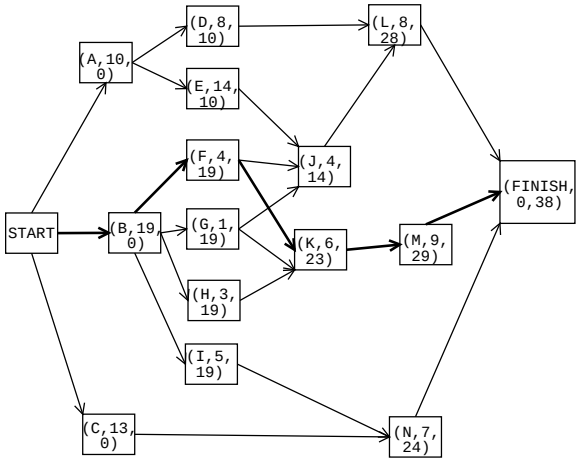


Figure 14: Project management example

Binary knapsack problem

Problem: Given a set of items, each with a weight and value (cost), determine the subset of items with maximum total weight and total cost less than a given budget.

The binary knapsack problem's mathematical programming formulation is as follows.

$$\begin{array}{rcl} \max & \displaystyle\sum_{j=1}^{n} w_j x_j & \\ & \displaystyle\sum_{j=1}^{n} v_j x_j & \leq \quad B \\ & x_j & \in \quad \{0,1\} \end{array}$$

The binary knapsack problem is a hard problem. However it can be solved as a longest path problem on a DAG as follows.

Let $f_i(q)$ be the maximum weight achievable when considering the first i items and a budget of q, that is,

$$\begin{aligned} f_i(q) = \max \quad & \sum_{j=1}^{i} w_j x_j \\ & \sum_{j=1}^{i} v_j x_j \quad \leq \quad q \\ & \qquad x_j \quad \in \quad \{0,1\} \end{aligned}$$

Note that we are interested in finding $f_n(B)$. Additionally, note that the $f_i(q)$'s are related with the following dynamic programming equation:

$$f_{i+1}(q) = \max\{f_i(q), f_i(q - v_{i+1}) + w_{i+1}\}.$$

The above equation can be interpreted as follows. By the principle of optimality, the maximum weight I can obtain, when considering the first $i+1$ items will be achieved by either:

1. Not including the $i+1$ item in my selection. In which case my total weight will be equal to $f_i(q)$, i.e. equal to the maximum weight achievable with a budget of q when only considering only the first i items; or
2. Including the $i+1$ item in my selection. In which case my total weight will be equal to $f_i(q - v_{i+1}) + w_{i+1}$, i.e. equal to the maximum weight achievable when considering the first i items with a budget of $q - v_{i+1}$ plus the weight of this $i+1$ object.

We also have the the boundary conditions:

$$f_1(q) = \begin{cases} 0 & 0 \leq q < v_1 \\ w_1 & B \geq q \geq v_1 \end{cases}.$$

The graph associated with the knapsack problem with 4 items, $w = (6, 8, 4, 5)$, $v = (2, 3, 4, 4)$, and B=12 is given in Figure 15. (The figure was adapted from: Trick M., A dynamic programming approach for consistency and propagation of knapsack constraints.)

Note that in general a *knapsack graph* will have $O(nB)$ nodes and $O(nB)$ edges (each node has at most two incoming arcs), it follows that we can solve the binary knapsack problem in time $O(nB)$.

remark The above running time is **not** polynomial in the length of the input. This is true because the number B is given using only $\log B$ bits. Therefore the size of the input for the knapsack problem is $O(n \log B + n \log W)$ (where W is the maximum w_i). Finally, since, $B = 2^{\log B}$, then a running time of $O(nB)$ is really exponential in the size of the input.

8.7 Dijkstra's algorithm

Dijkstra's algorithm is a special purpose algorithm for finding shortest paths from a single source, s, to all other nodes in a graph with non-negative edge weights.

In words, Dijkstra's algorithm assigns distance labels (from node s) to all other nodes in the graph. Node labels are either *temporary* or *permanent*. Initially all nodes have temporary labels.

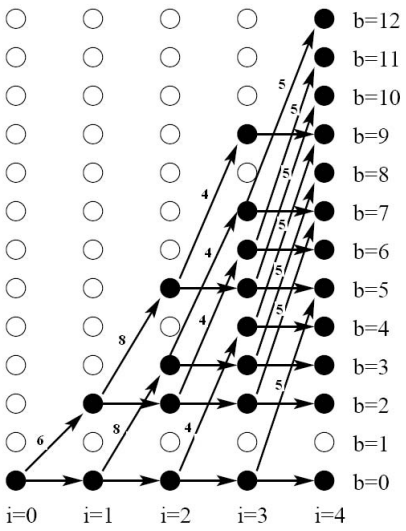


Figure 15: Graph formulation of a binary knapsack problem.

At any iteration, a node with the least distance label is marked as permanent, and the distances to its successors are updated. This is continued until no temporary nodes are left in the graph.

We give the pseudocode for Dijkstra's algorithm below. Where P is the set of nodes with permanent labels, and T of *temporarily labeled nodes.*

```
begin
N+(i) := {j|(i,j) ∈ A};
P := {1}; T := V \ {1};
d(1) := 0 and pred(1) := 0;
d(j) := c_1j and pred(j) := 1 for all j ∈ A(1),
and d(j) := ∞ otherwise;
while P ≠ V do
      begin
      (Node selection, also called FINDMIN)
      let i ∈ T be a node for which d(i) = min{d(j) : j ∈ T};
      P := P ∪ {i}; T := T \ {i};
      (Distance update)
      for each j ∈ N+(i) do
          if d(j) > d(i) + c_ij then
         d(j) := d(i) + c_ij and pred(j) := i;
      end
end
```

Theorem 8.7 (The correctness of Dijkstra's algorithm)**.** *Once a node joins P its label is the shortest path label.*

Proof. At each iteration the nodes are partitioned into subsets P and T. We will prove by induction on the size of P that the label for each node $i \in P$ is the shortest distance from node 1.
Base: When $|P| = 1$, the only node in P is s. It is correctly labeled with a distance 0.
Inductive step: Assume for $|P| = k$ and prove for $|P| = k + 1$.

Suppose that node i with $d(i) = min\{d(j) : j \in T\}$, was added to P, but its label $d(i)$ is not the shortest path label. Then there should be some nodes in T along the shortest path 1 to i. Let j be the first node in T on the shortest path from 1 to i. Then the length of shortest path from 1 to $i = d(j) + c_{ji}$. Since $d(i)$ is not the shortest label, $d(j) + c_{ji} < d(i)$, and since c_{ji} is positive, we have $d(j) < d(i)$. This contradicts that $d(i)$ is the minimum of $d(j) \in T$. □

We next prove two invariants that are preserved during Dijkstra's algorithm execution.

Proposition 8.8. *The label for each $j \in T$ is the shortest distance from s such that all nodes of the path are in P (i.e. it would be the shortest path if we eliminated all arcs with both endpoints in T).*

Proof. We again use induction on the size of P.
Base: When $|P| = 1$, the only node in P is s. Also, all distance labels are initialized as follows: $d(j) := c_{sj}$ for all $j \in A(1)$. Thus the labels are indeed the shortest distance from s, using allowing only the path (s, j).
Inductive step: Assume for $|P| = k$ and prove for $|P| = k + 1$.
After a node i in T is labeled permanently, the distance labels in $T \setminus \{i\}$ might decrease. After permanently labeling node i, the algorithm sets $d(j) = d(i) + c_{ij}$ if $d(j) > d(i) + c_{ij}$, for $(i, j) \in A$. Therefore, after the distance update, by the inductive hypothesis, the path from source to j satisfies $d(j) \leq d(i) + c_{ij}$, $\forall\, (i, j) \in A$ s.t $i \in P$. Thus, the distance label of each node in $T \setminus \{i\}$ is the length of the shortest path subject to the restriction that each every node in the path must belong to $P \bigcup \{i\}$. □

Proposition 8.9. *The labels of nodes joining P can only increase in successive iterations.*

Proof. We again use induction on the size of P.
Base: When $|P| = 1$, the only node in P is s, which has a label 0. Since all costs are nonnegative, all other nodes will have labels ≥ 0.
Inductive step: Assume for $|P| = k$ and prove for $|P| = k + 1$.
Assume by contradiction that node i with $d(i) = min\{d(j) : j \in T\}$, was added to P, but in a later iteration node j is added to P having a label $d(j) < d(i)$. This contradicts the Theorem 8.7 since if the graph had a zero-cost arc (j, i), then the shortest distance from s to i would be $d(j)$ $(< d(i)$ which was the label of node i when added to the permanent set). □

Complexity Analysis
There are two major operations in Dijkstra's algorithm :
Find minimum, which has $O(n^2)$ complexity; and
Update labels, which has $O(m)$ complexity.
Therefore, the complexity of Dijkstra's algorithm is $= O(n^2 + m)$.

There are several implementations that improve upon this running time.

One improvement uses *Binary Heaps*: whenever a label is updated, use binary search to insert that label properly in the sorted array. This requires $O(\log n)$. Therefore, the complexity of finding the minimum label in temporary set is $O(m \log n)$, and complexity of the algorithm is $O(m + m \log n) = O(m \log n)$.

Another improvement is *Radix Heap* which has complexity $O(m+n\sqrt{\log C})$ where $C = max_{(i,j)\in A} c_{ij}$. But this is not a strongly polynomial complexity.

Currently, the best *strongly polynomial* implementation of Dijkstra's algorithm uses *Fibonacci Heaps* and has complexity $O(m + n \log n)$. Since Fibonacci heaps are hard to program, this implementation is not used in practice.

It is important to stress that Dijkstra's algorithm does not work correctly in the presence of negative edge weights. In this algorithm once a node is included in the permanent set it will not be checked later again, but because of the presence of negative edge that is not correct. The problem is that the distance label of a node in the permanent set might be reduced after its inclusion in the permanent set (when considering negative cost arcs).

See Figure 16 for an example such that Dijkstra's algorithm does not work in presence of negative edge weight. Using the algorithm, we get $d(3) = 3$. But the actual value of $d(3)$ is 2.

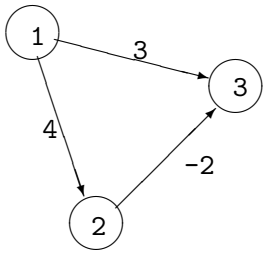


Figure 16: Network for which Dijkstra's algorithm fails

8.8 Bellman-Ford algorithm for single source shortest paths

This algorithm can be used with networks that contain negative weighted edges. Therefore it detects the existence of negative cycles. In words, the algorithm consists of n pulses. (pulse = update of labels along all m arcs). (Henceforth we will assume that we want to find the shortest paths from node 1 to all other nodes.)

Input: A (general) graph $G = (V, A)$, and two nodes s and t.
Output: The shortest path from s to t.
Pseudocode
$Initialize : d(1) := 0, d(i) := \infty, i = 2, \ldots, n$
For $i = 1, \ldots, n$ **do**
$Pulse : \forall\ (i, j) \in A, d(j) := min\{d(j), d(i) + C_{ij}\}$

To prove the algorithm's correctness we will use the following lemma.

Lemma 8.10. *If there are no negative cost cycles in the network $G = (N, A)$, then there exists a shortest path from s to any node i which uses at most $n - 1$ arcs.*

Proof. Suppose that G contains no negative cycles. Observe that at most $n - 1$ arcs are required to construct a path from s to any node i. Now, consider a path, P, from s to i which traverses a cycle.

$$P = s \to i_1 \to i_2 \to \ldots \to (i_j \to i_k \to \ldots \to i_j) \to i_\ell \to \ldots \to i.$$

Since G has no negative length cycles, the length of P is no less than the length of $\bar{P}$ where

$$\bar{P} = s \to i_1 \to i_2 \to \ldots \to i_j \to i_k \to \ldots \to i_\ell \to \ldots \to i.$$

Thus, we can remove all cycles from P and still retain a shortest path from s to i. Since the final path is acyclic, it must have no more than $n-1$ arcs. □

Theorem 8.11 (Invariant of the algorithm)**.** *After pulse k, all shortest paths from s of length k (in terms of number of arcs) or less have been identified.*

Proof. Notation : $d_k(j)$ is the label $d(j)$ at the end of pulse k.

By induction. For $k = 1$, obvious. Assume true for k, and prove for $k+1$.

Let $(s, i_1, i_2, ..., i_\ell, j)$ be a shortest path to j using $k+1$ or fewer arcs.

It follows that $(s, i_1, i_2, ..., i_\ell)$ is a shortest path from s to i_ℓ using k or fewer arcs.

After pulse $k+1$, $d_{k+1}(j) \leq d_k(i_\ell) + c_{i_\ell,j}$.

But $d_k(i_\ell) + c_{i_\ell,j}$ is the length of a shortest path from s to j using $k+1$ or fewer arcs, so we know that $d_{k+1}(j) = d_k(i_\ell) + c_{i_\ell,j}$.

Thus, the $d()$ labels are correct after pulse $k+1$. □

Note that the shortest paths identified in the theorem are not necessarily *simple*. They may use negative cost cycles.

Theorem 8.12. *If there exists a negative cost cycle, then there exists a node j such that $d_n(j) < d_{n-1}(j)$ where $d_k(i)$ is the label at iteration k.*

Proof. Suppose *not*.

Let $(i_1 - i_2 - ... - i_k - i_1)$ be a negative cost cycle.

We have $d_n(i_j) = d_{n-1}(i_j)$, since the labels never increase.

From the optimality conditions, $d_n(i_{r+1}) \leq d_{n-1}(i_r) + C_{r,r+1}$ where $r = 1,...,k\text{-}1$, and $d_n(i_1) \leq d_{n-1}(i_k) + C_{i_k,i_1}$.

These inequalities imply $\sum_{j=1}^{k} d_n(i_j) \leq \sum_{j=1}^{k} d_{n-1}(i_j) + \sum_{j=1}^{k} C_{i_{j-1},i_j}$.

But if $d_n(i_j) = d_{n-1}(i_j)$, the above inequality implies $0 \leq \sum_{j=1}^{k} C_{i_{j-1},i_j}$.

$\sum_{j=1}^{k} C_{i_{j-1},i_j}$ is the length of the cycle, so we get an inequality which says that 0 is less than a negative number which is a contradiction. □

Corollary 8.13. *The Bellman-Ford algorithm identifies a negative cost cycle if one exists.*

By running n pulses of the Bellman-Ford algorithm, we either detect a negative cost cycle (if one exists) or find a shortest path from s to all other nodes i. Therefore, finding a negative cost cycle can be done in polynomial time. Interestingly, finding the most negative cost cycle is *NP*-hard (reduction from Hamiltonian Cycle problem).

Remark Minimizing mean cost cycle is also polynomially solvable.

Complexity Analysis

Complexity = $O(mn)$ since each pulse is $O(m)$, and we have n pulses.

Note that it suffices to apply the pulse operations only for arcs (i, j) where the label of i changed in the previous pulse. Thus, if no label changed in the previous iteration, then we are done.

8.9 Floyd-Warshall algorithm for all pairs shortest paths

The Floyd-Warshall algorithm is designed to solve all pairs shortest paths problems for graphs with negative cost edges. As all algorithms for shortest paths on general graphs, this algorithm will detect negative cost cycles.

In words, the algorithm maintains a matrix (d_{ij}) such that at iteration k, d_{ij} is the shortest path from i to j using nodes $1, 2, ...k$ as intermediate nodes. After the algorithm terminates, assuming that no negative cost cycle is present, the shortest path from nodes i to j is d_{ij}.

Pseudocode

Input: An $n \times n$ matrix $[c_{ij}]$
Output: An $n \times n$ matrix $[d_{ij}]$ is the shortest distance
from i to j under $[c_{ij}]$
An $n \times n$ matrix $[e_{ij}]$ is a node in the path from
i to j.

begin
for all $i \neq j$ **do** $d_{ij} := c_{ij}$;
for $i = 1, \ldots, n$ **do** $d_{ii} := \infty$;
for $j = 1, \ldots, n$ **do**
for $i = 1, \ldots, n, i \neq j$, **do**
for $k = 1, \ldots, n, k \neq j$, **do**
$d_{ik} := \min\{d_{ik}, d_{ij} + d_{jk}\}$

$$e_{ik} := \begin{cases} j & \text{if } d_{ik} > d_{ij} + d_{jk} \\ e_{ik} & \text{otherwise} \end{cases}$$

end

The main operation in the algorithm is: *Pulse* j : $d_{ik} = \min(d_{ik}, d_{ij} + d_{jk})$. This operation is sometimes referred to as a *triangle operation* (see Figure 17).

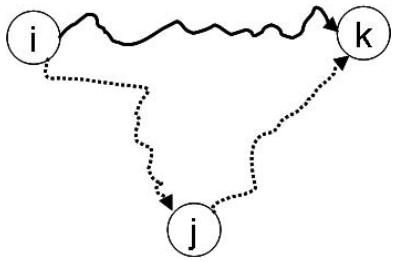


Figure 17: Triangle operation: Is the (dotted) path using node j as intermediate node shorter than the (solid) path without using node j?

Invariant property of Floyd-Warshall Algorithm:
After iteration k, d_{ij} is the shortest path distance from i to j involving a subset of nodes in $\{1, 2, \ldots, k\}$ as intermediate nodes.

Complexity Analysis
At each iteration we consider another node as intermediate node $\rightarrow O(n)$ iterations.
In each iteration we compare n^2 triangles for n^2 pairs $\rightarrow O(n^2)$.
Therefore complexity of the Floyd-Warshall algorithm $= O(n^3)$.

See the handout titled *All pairs shortest paths.*

At the initial stage, $d_{ij} = c_{ij}$ if there exist an arc between node i and j
$d_{ij} = \infty$ otherwise
$e_{ij} = 0$
First iteration : $d_{24} \leftarrow \min(d_{24}, d_{21} + d_{14}) = 3$
$e_{24} = 1$
Update distance label
Second iteration : $d_{41} \leftarrow \min(d_{41}, d_{42} + d_{21}) = -2$
$d_{43} \leftarrow \min(d_{43}, d_{42} + d_{23}) = -3$
$d_{44} \leftarrow \min(d_{44}, d_{42} + d_{24}) = -1$
$e_{41} = e_{43} = e_{44} = 2$
No other label changed

Note that we found a negative cost cycle since the diagonal element of matrix $D, d_{44} = -1 \leq 0$. Negative d_{ii} means there exists a negative cost cycle since it simply says that there exists a path of negative length from node i to itself.

8.10 D.B. Johnson's algorithm for all pairs shortest paths

See the handout titled *All pairs shortest paths - section 1.2.*

This algorithm takes advantage of the idea that if all arc costs were nonnegative, we can find all pairs shortest paths by solving $O(n)$ single source shortest paths problems using Dijkstra's algorithm. The algorithm converts a network with negative-cost arcs to an equivalent network with nonnegative-cost arcs. After this transformation, we can use Dijkstra's algorithm to solve n SSSP (and get our APSP).

D.B. Johnson's algorithm

1. Add a node s to the graph with $c_{sj} = 0$ for all nodes $j \in V$.
2. Use Bellman-Ford algorithm to find the shortest paths from s to every other node. If a negative cost cycle is detected by the algorithm, then terminate at this step. Otherwise, let $d(i)$ be the shortest distance from s to node i.
3. Remove s from the graph, and convert the costs in the original graph to nonnegative costs by setting $c'_{ij} = c_{ij} + d(i) - d(j)$.
4. Apply Dijkstra's algorithm n times to the graph with weights c'.

The correctness of the algorithm follows from the claims below.

Claim 8.14. *In step 1, adding node s to the graph does not create or remove negative cost cycles.*

Proof. Since s only has outgoing arcs, then it cannot be part of any cycle; similarly the added arcs cannot be part of any cycle. Therefore we have not created any more cycles in G. Finally, if G contained a negative cycle, this cycle remains unchanged after adding s to G. □

Claim 8.15. *In step 3,* $c'_{ij} \geq 0 \forall (i,j) \in A$.

Proof. Since $d(i)$ are shortest path labels. They must satisfy $c_{ij} + d(i) \geq d(j)$ (Proposition 2 in Lecture 9). This implies that $c'_{ij} = c_{ij} + d(i) - d(j) \geq 0$ □

Let $d_c^P(s,t)$ be the distance of a path P from s to t with the costs c.

Claim 8.16. *For every pair of nodes s and t, and a path P, $d_c^P(s,t) = d_{c'}^P(s,t) - d(s) + d(t)$. That is, minimizing $d_c(s,t)$ is equivalent to minimizing $d_{c'}(s,t)$.*

Proof. Let the path from s to t, be $P = (s, i_1, i_2, .., i_k, t)$.

The distance along the path satisfies,

$$d_{c'}^P(s,t) = c_{si_1} + d(s) - d(i_1) + c_{si_2} + d(i_1) - d(i_2) \ldots + c_{si_k} + d(i_k) - d(t) = d_c^P(s,t) + d(s) - d(t).$$

Note that the $d(i)$ terms cancel out except for the first and last terms. Thus the length of all paths from s to t, when using costs c' instead of costs c, change by the constant amount $d(s) - d(t)$.

Therefore, the distances of path P when using c and c' are related as follows:

$$d_{c'}^P(s,t) = d_c^P(s,t) + d(s) - d(t).$$

Thus, $\min_P d_{c'}^P(s,t) = \min_P d_c^P(s,t)+$ constant. Hence minimizing $d_{c'}(s,t)$ is equivalent to minimizing $d_c(s,t)$. □

Complexity Analysis
step 1: Add s and n arcs $\rightarrow O(n)$ step 2: Bellman-Ford $\rightarrow O(mn)$.
step 3: Updating c'_{ij} for m arcs $\rightarrow O(m)$.
step 4: Dijkstra's algorithm n times $\rightarrow O(n(m + n \log n))$.
Total complexity: $O(n(m + n \log n))$.

So far this has been the best complexity established for the all-pairs shortest paths problem (on general graphs).

8.11 Matrix multiplication algorithm for all pairs shortest paths

The algorithm

Consider a matrix D with $d_{ij} = c_{ij}$ if there exists an arc from i to j, ∞ otherwise.
Let $d_{ii} = 0$ for all i. Then we define the "dot" operation as
$D_{ij}^{(2)} = (D \odot D^T)_{ij} = \min\{d_{i1} + d_{1j}, \ldots, d_{in} + d_{nj}\}$.

$D_{ij}^{(2)}$ represents the shortest path with 2 or fewer edges from i to j .
$D_{ij}^{(3)} = (D \odot D^{(2)}T)_{ij}$ = shortest path with number of edges ≤ 3 from i to j
$\vdots$
$D_{ij}^{(n)} = (D \odot D^{(n-1)T})_{ij}$ = shortest path with number of edges $\leq n$ from i to j

Complexity Analysis
One matrix "multiplication" (dot operation) $\rightarrow O(n^3)$
n multiplications leads to the total complexity $O(n^4)$

By doubling, we can improve the complexity to $O(n^3 log_2 n)$. Doubling is the evaluation of $D_{ij}^{(n)}$ from its previous matrices that are powers of two.

"Doubling": Improving Matrix Multiplication

Improving complexity by Doubling (binary powering):
$D^{(4)} = D^{(2)} \odot D^{(2)T}$
$D^{(8)} = D^{(4)} \odot D^{(4)T}$
$\vdots$
$D^{(2^k)} = D^{(2^{k-1})} \odot D^{(2^{k-1})T}$
Stop once k is such that $2^k > n$ (i.e. need only calculate up to $k = \lceil log_2 n \rceil$).
Then, $D^{(n)}$ can be found by "multiplying" the appropriate matrices (based on the binary representation of n) that have been calculated via doubling. Or we can compute $D^{(2^k)}$, where $k = \lceil log_2 n \rceil$ and it will be equal to $D^{(n)}$ since $D^{(n)} = D^{(n)} \odot D^T$ (because the shortest path may go through at most n nodes).
$\Rightarrow$ Complexity of algorithm with *doubling* is $(n^3 log_2 n)$.

Back-tracking:
To find the paths, construct an $'e'$ matrix as for Floyd-Warshall's algorithm where $e_{ij}^{(k)} = p$ if

$$\min(d_{i1}^{(k-1)} + d_{1j}^{(k-1)}, \ldots, d_{in}^{(k-1)} + d_{nj}^{(k-1)}) = d_{ip}^{(k-1)} + d_{pj}^{(k-1)}.$$

p is then an intermediate node on the shortest path of length $\leq k$ from i to j.

Identifying negative cost cycles:
This is left as a question to class. Suppose there is a negative cost cycle. Then how is it detected if we only have powers of 2 computations of the matrix? The problem is to show that if there is a negative cost cycle then starting at some iteration $\leq n$ the distance labels of some nodes must go down at each iteration. What are these nodes?

Seidel's algorithm for Unit Cost Undirected graph:
(1) Seidel has used the matrix multiplication approach for the all pairs shortest paths problem in an undirected graph. Seidel's algorithm only works when the cost on all the arcs is 1. It uses actual matrix multiplication to find all pairs shortest paths. The running time of Seidel's algorithm is $O(M(n) \log n)$, where $M(n)$ is the time required to multiply two $n \times n$ matrices.
(2) Currently, the most efficient (in terms of asymptotic complexity) algorithm known for matrix multiplication is $O(n^{2.376})$. However, there is a huge constant which is hidden by the complexity notation, so the algorithm is not used in practice.

8.12 Why finding shortest paths in the presence of negative cost cycles is difficult

Suppose we were to apply a minimum cost flow formulation to solve the problem of finding a shortest path from node s to node t in a graph which contains negative cost cycles. We could give a capacity of 1 to the arcs in order to guarantee a bounded solution. This is not enough, however,

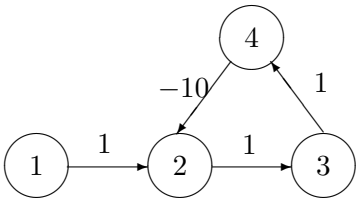


Figure 18: Shortest Path Problem With Negative Cost Cycles

to give us a correct solution. The problem is really in ensuring that we get a *simple* path as a solution.

Consider the network in Figure 18. Observe first that the shortest *simple* path from node 1 to node 2 is the arc $(1, 2)$. If we formulate the problem of finding a shortest path from 1 to 2 as a MCNF problem, we get the following linear program:

$$\begin{array}{lll} \min & x_{12} + x_{23} + x_{34} - 10x_{42} & \\ \text{s.t.} & x_{12} & = \ 1 \\ & x_{23} - x_{12} - x_{42} & = \ -1 \\ & x_{34} - x_{23} & = \ 0 \\ & x_{42} - x_{34} & = \ 0 \\ & 0 \leq x_{ij} \leq 1 & \forall (i,j) \in A. \end{array}$$

Recall that the interpretation of this formulation is that $x_{ij} = 1$ in an optimal solution means that arc (i, j) is on a shortest path from node 1 to node 2 and the path traced out by the unit of flow from node 1 to node 2 is a shortest path. In this case, however, the optimal solution to this LP is $x_{12} = x_{23} = x_{34} = x_{42} = 1$ which has value -7. Unfortunately, this solution does not correspond to a simple path from node 1 to node 2.

9 Maximum flow problem

9.1 Introduction

See the handout titled *Travelers Example: Seat Availability.*

The *Maximum Flow* problem is defined on a directed network $G = (V, A)$ with capacities u_{ij} on the arcs, and no costs. In addition two nodes are specified, a source node, s, and sink node, t. The objective is to find the maximum flow possible between the source and sink while satisfying the arc capacities. (We assume for now that all lower bounds are 0.)

Definitions:

- A *feasible flow* f is a flow that satisfied the flow-balance and capacity constraints.
- A *cut* is a partition of the nodes (S, T), such that $S \subseteq V$, $s \in S$, $T = V \setminus S$, $t \in T$.
- *Cut capacity:* $U(S, T) = \sum_{i \in S} \sum_{j \in T} u_{ij}$. (**Important:** note that the arcs in the cut are **only** those that go from S to T.)

Let $|f|$ be the total amount of flow out of the source (equivalently into the sink). It is easy to

observe that any feasible flow satisfies

$$|f| \leq U(S,T) \tag{13}$$

for any (S,T) cut. This is true since all flow goes from s to t, and since $s \in S$ and $t \in T$ (by definition of a cut), then all the flow must go through the arcs in the (S,T) cut. Inequality 13 is a special case of the *weak duality theorem* of linear programming.

The following theorem, which we will establish algorithmically, can be viewed as a special case of the strong duality theorem in linear programming.

Theorem 9.1 (Max-Flow Min-Cut)**.** *The value of a maximum flow is equal to the capacity of a cut with minimum capacity among all cuts. That is,*

$$\max |f| = \min U(S,T)$$

Next we introduce the terminology needed to prove the Max-flow/Min-cut duality and to give us an algorithm to find the max flow of any given graph.

Residual graph: The residual graph, $G_f = (V, A_f)$, with respect to a flow f, has the following arcs:

- forward arcs: $(i,j) \in A_f$ if $(i,j) \in A$ and $f_{ij} < u_{ij}$. The residual capacity is $u^f_{ij} = u_{ij} - f_{ij}$. The residual capacity on the forward arc tells us how much we can increase flow on the original arc $(i,j) \in A$.
- reverse arcs: $(j,i) \in A_f$ if $(i,j) \in A$ and $f_{ij} > 0$. The residual capacity is $u^f_{ji} = f_{ij}$. The residual capacity on the reverse arc tells us how much we can decrease flow on the original arc $(i,j) \in A$.

Intuitively, residual graph tells us how much **additional** flow can be sent through the original graph with respect to the given flow. This brings us to the notion of an augmenting path.

In the presence of lower bounds, $\ell_{ij} \leq f_{ij} \leq u_{ij},\ (i,j) \in A$, the definition of forward arc remains, whereas for reverse arc, $(j,i) \in A_f$ if $(i,j) \in A$ and $f_{ij} > \ell_{ij}$. The residual capacity is $u^f_{ji} = f_{ij} - \ell_{ij}$.

9.2 Linear programming duality and max flow min cut

Given $G = (N, A)$ and capacities u_{ij} for all $(i,j) \in A$:
Consider the formulation of the maximum flow problem on a general graph. Let x_{ij} be a variable denoting the flow on arc (i,j).

$$\begin{array}{ll} \text{Max} & x_{ts} \\ \text{subject to} & \sum_i x_{ki} - \sum_j x_{jk} = 0 \quad k \in V \\ & 0 \leq x_{ij} \leq u_{ij} \quad \forall (i,j) \in A. \end{array}$$

For dual variables we let $\{z_{ij}\}$ be the nonnegative dual variables associated with the capacity upper bounds constraints, and $\{\lambda_i\}$ the variables associated with the flow balance constraints. (I also multiply the flow balance constraints by -1, so they represent $\text{Inflow}_i - \text{Outflow}_i = 0$.)

$$\begin{array}{ll} \text{Min} & \sum_{(i,j)\in A} u_{ij} z_{ij} \\ \text{subject to} & z_{ij} - \lambda_i + \lambda_j \geq 0 \quad \forall (i,j) \in A \\ & \lambda_s - \lambda_t \geq 1 \\ & z_{ij} \geq 0 \qquad \forall (i,j) \in A. \end{array}$$

The dual problem has an infinite number of solutions: if (λ^*, z^*) is an optimal solution, then so is $(\lambda^* + C, z^*)$ for any constant δ. To avoid that we set $\lambda_t = 0$ (or to any other arbitrary value). Observe now that with this assignment there is an optimal solution with $\lambda_s = 1$ and a partition of the nodes into two sets: $S = \{i \in V | \lambda_i = 1\}$ and $\bar{S} = \{i \in V | \lambda_i = 0\}$.

The complementary slackness condition state that the primal and dual optimal solutions x^*, λ^*, z^* satisfy,

$$x^*_{ij} \cdot [z^*_{ij} - \lambda^*_i + \lambda^*_j] = 0$$
$$[u_{ij} - x^*_{ij}] \cdot z^*_{ij} = 0.$$

In an optimal solution $z^*_{ij} - \lambda^*_i + \lambda^*_j = 0$ except for the arcs in $(\bar{S}, S)$, so the first set of complementary slackness conditions provides very little information on the primal variables $\{x_{ij}\}$. Namely, that the flows are 0 on the arcs of $(\bar{S}, S)$. As for the second set, $z^*_{ij} = 0$ on all arcs other than the arcs in the cut $(S, \bar{S})$. So we can conclude that the cut arcs are saturated, but derive no further information on the flow on other arcs.

The only method known to date for solving the minimum cut problem requires finding a maximum flow first, and then recovering the cut partition by finding the set of nodes reachable from the source in the residual graph (or reachable from the sink in the reverse residual graph). That set is the source set of the cut, and the recovery can be done in linear time in the number of arcs, $O(m)$.

On the other hand, if we are given a minimum cut, there is no efficient way of recovering the flow values on each arc other than essentially solving from scratch. The only information given by the minimum cut, is the value of the maximum flow and the fact that the arcs on the cut are saturated. Beyond that, the flows have to be calculated with the same complexity as would be required without the knowledge of the minimum cut.

This asymmetry implies that it may be easier to solve the minimum cut problem than to solve the maximum flow problem. Still, no minimum cut algorithm has ever been discovered, in the sense that every so-called s, t minimum cut algorithm known, computes first the maximum flow.

9.3 Applications

9.3.1 Hall's theorem

Hall discovered this theorem about a necessary and sufficient condition for the existence of a perfect matching independently of the max flow min cut theorem. However the max flow min cut theorem can be used as a quick alternative proof to Hall's theorem.

Theorem 9.2 (Hall)**.** *Given a bipartite graph, $B = (U \cup V, E)$ where $|U| = |U| = n$.*
There is a perfect matching in B iff $\forall X \subseteq U$, $|N(X)| \geq |X|$,
where $N(X)$ is the set of neighbors of X, $N(X) \subseteq V$.

Proof.
($\Rightarrow$) If there is a perfect matching, then for every $X \subseteq U$ the set of its neighbors $N(X)$ contains at least all the nodes matched with X. Thus, $|N(X)| \geq |X|$.

($\Leftarrow$) We construct a flow graph by adding: a source s, a sink t, unit-capacity arcs from s to all nodes in U, and unit-capacity arcs from all nodes in V to t. Direct all arcs in the original graph G from U to V, and set the capacity of these arcs to ∞ (See Figure 19).

Assume that $|N(X)| \geq |X|$, $\forall\ X \subseteq U$. We need to show that there exists a perfect matching. Consider a finite cut (S, T) in the flow graph defined. Let $X = U \cap S$. We note that $N(X) \subseteq S \cap V$, else the cut is not finite.

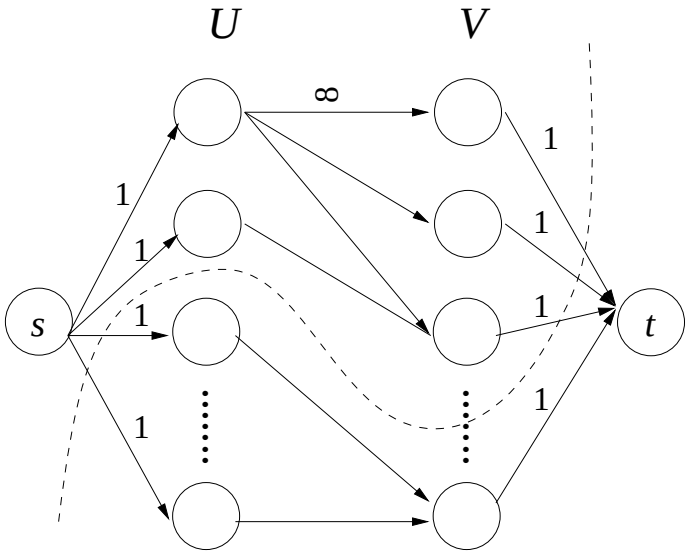


Figure 19: Graph for Hall's Theorem

Since $|N(X)| \geq |X|$, the capacity of the any (S, T) cut satisfies,

$$U(S,T) \geq |N(X)| + n - |X| \geq n.$$

Also, because of the construction of the network, $U(S,T) \leq n$. Hence $U(S,T) = n$.

Now, for (S,T) a minimum cut,

$$n \leq U(S,T) = \text{ max flow value } = \text{ size of matching } \leq n.$$

It follows that all the inequalities must be satisfied as equalities, and the matching size is n. □

Define, **Discrepancy** $= -\min_X \{|N(X)| - |X|\} = \max_X \{|X| - |N(X)|\}$, $X \subseteq U$
Since $U(S,T) = n + |\Gamma(X)| - |X|$, where $X = U \cap S$,

$$\min U(S,T) = n + \min\{|\Gamma(X)| - |X|\}.$$

Lemma 9.3. *For a bipartite graph with discrepancy D, there is a matching of size $n - D$*

9.3.2 The selection problem

Suppose you are going on a hiking trip. You need to decide what to take with you. Many individual items are useless unless you take something else with them. For example, taking a bottle of wine without an opener does not make too much sense. Also, if you plan to eat soup there, for example, you might want to take a set of few different items: canned soup, an opener, a spoon and a plate. The following is the formal definition of the problem.

Given a set of items $J = \{1, \ldots, n\}$, a cost for each item c_j, a collection of sets of items $S_i \subseteq J$ for $i = 1, \ldots, m$, and a benefit for each set b_i. We want to maximize the net benefit (= total benefit - total cost of items) selected.

We now give the mathematical programming formulation:

$$\begin{aligned} \max \quad & \sum_{j=1}^{m} b_i x_i - \sum_{i=1}^{n} c_j y_j \\ & x_i \leq y_j \quad \text{for } i = 1, \ldots, m, \forall j \in S_i \\ & x_i \in \{0,1\} \quad \text{for } i = 1, \ldots, m \\ & y_j \in \{0,1\} \quad \text{for } j = 1, \ldots, n. \end{aligned}$$

Where,

$$x_i = \begin{cases} 1 & \text{if set } j \text{ is selected} \\ 0 & \text{otherwise} \end{cases}$$

$$y_j = \begin{cases} 1 & \text{if item } j \text{ is included} \\ 0 & \text{otherwise} \end{cases}$$

We immediately recognize that the constraint matrix is totally unimodular (each row has one 1 and one -1). Thus when solving the LP relaxation of the problem we will get a solution for the above IP.

Nevertheless, we can do better than solving the LP. In particular we now show how the Selection Problem can be solved by finding a minimum *s-t* cut in an appropriate network.

First we make the following observation. The optimal solution to the selection problem will consist of a collection of sets, $S_j, j \in J$, and the union of their elements ($\bigcup S_j, j \in J$). In particular we will never pick "extra items" since this only adds to our cost. Feasible solutions with this structure are called *selections*.

As per the above discussion, we can limit our search to selections, and we can formulate the selection problem as follows. Find the collection of sets S such that:

$$\max_{S} \sum_{i \in S} b_i - \sum_{j \in \bigcup_{i \in S} S_i} c_j \tag{14}$$

The following definition will be useful in our later discussion. Given a directed graph $G = (V, A)$, a *closed set* is a set $C \subseteq V$ such that $u \in C \text{and} (u, v) \in A \implies v \in C$. That is, a closed set includes all of the successors of every node in C. Note that both $\emptyset$ and V are closed sets.

Now we are ready to show how we can formulate the selection problem as a minimum cut problem. We first create a bipartite graph with sets on one side, items on the other. We add arcs from each set to all of its items. (See figure 20.) Note that a selection in this graph is represented by a collection of set nodes and all of its successors. Indeed there is a one-to-one correspondence between selections and closed sets in this graph.

Next we transform this graph into a maxflow-mincut graph. We set the capacity of all arcs to infinity. We add a source, a sink, arcs from s to each set i with capacity b_i, and arcs from from each set j to t with capacity c_j. (See figure 21.)

We make the following observations. There is a one-to-one correspondence between finite cuts and selections. Indeed the source set of any finite (S, T) cut is a selection. (If $S_j \in S$ then it must be true that all of its items are also in S—otherwise the cut could not be finite.) Now we are ready to state our main result.

Theorem 9.4. *The source set of a minimum cut (S,T) is an optimal selection.*

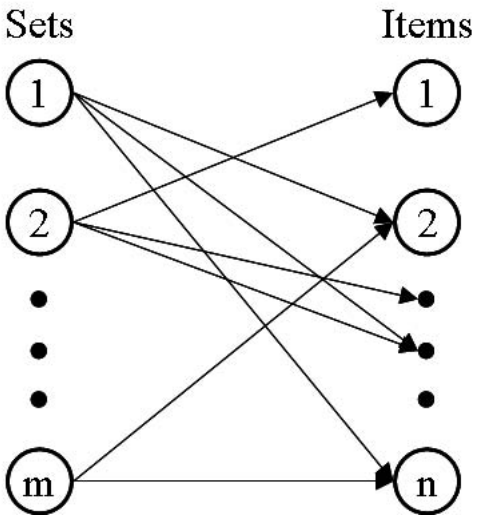


Figure 20: Representing selections as closed sets

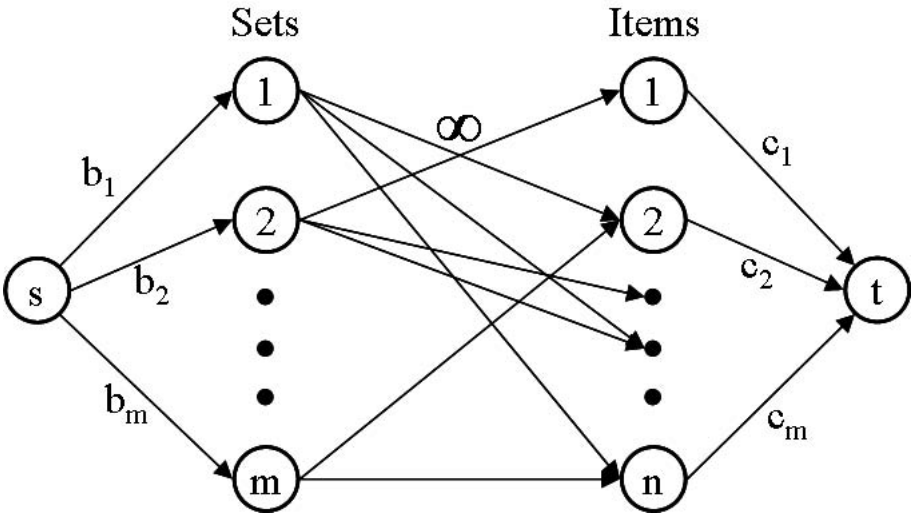


Figure 21: Formulation of selection problem as a min-cut problem

Proof.

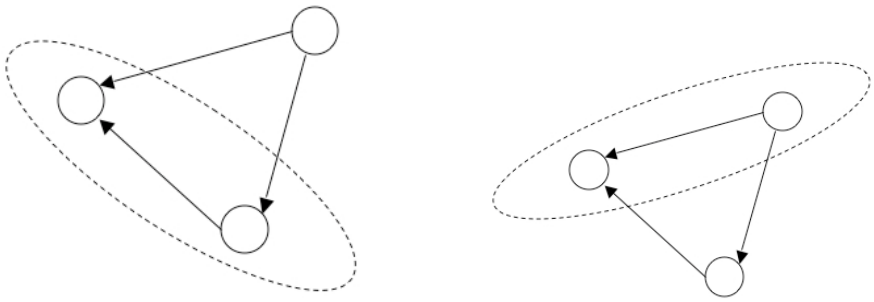
$$\begin{aligned}
\min_S U(S,T) &= \min \sum_{i\in T} b_i + \sum_{j\in S} c_j \\
&= \min \sum_{i=1}^{m} b_i - \sum_{i\in S} b_i + \sum_{j\in S} c_j \\
&= B + \min - \sum_{i\in S} b_i + \sum_{j\in S} c_j \\
&= B - \max \sum_{i\in S} b_i - \sum_{j\in S} c_j.
\end{aligned}$$

Where $B = \sum_{i=1}^{m} b_i$. □

9.3.3 The maximum closure problem

We begin by defining a closed set on a graph.

Definition 9.5. *Given a directed graph $G = (V, A)$, a subset of the nodes $D \subset V$ is* closed, *if for every node in D, its successors are also in D.*



(a) Closed set (b) Not a closed set

Consider a directed graph $G = (V, A)$ where every node $i \in V$ has a corresponding weight w_i. The *maximum closure problem* is to find a closed set $V' \subseteq V$ with maximum total weight. That is, the maximum closure problem is:

Problem Name: *Maximum closure*

Instance: *Given a directed graph $G = (V, A)$, and node weights (positive or negative) b_i for all $i \in V$.*

Optimization Problem: *find a closed subset of nodes $V' \subseteq V$ such that $\sum_{i \in V'} b_i$ is maximum.*

We can formulate the maximum closure problem as an integer linear program (ILP) as follows.

$$\begin{aligned} \max \quad & \sum_{i \in v}^{n} w_i x_i \\ \text{s.t.} \quad & x_i \leq x_j \quad \forall (i,j) \in A \\ & x_i \in \{0,1\} \quad \forall i \in V, \end{aligned}$$

where x_i be a binary variable that takes the value 1 if node i is in the maximum closure, and 0 otherwise. The first set of constraints force the requirement that for every node i included in the set, its successor is also in the set. Observe that since every row has at most one 1 and one -1, the constraint matrix is totally unimodular (TUM). Specifically, this structure indicates that the problem is a dual of a flow problem.

Apparently, Johnson [Jo1] seems to be the first researcher who demonstrated that the maximum closure problem is equivalent to the selection problem (max closure on bipartite graphs), and that the selection problem is solvable by max flow algorithm. Picard [Pic1], demonstrated that a minimum cut algorithm on a related graph, solves the maximum closure problem.

Let $V^+ \equiv \{j \in V | w_j > 0\}$, and $V^- \equiv \{j \in V | w_j < 0\}$. We construct an s,t-graph G_{st} as follows. Given the graph $G = (V, A)$ we set the capacity of all arcs in A equal to ∞. We add a source s, a sink t, arcs from s to all nodes $i \in V^+$ (with capacity $u_{s,i} = w_i$), and arcs from all nodes

$j \in V^-$ to t (with capacity $u_{j,t} = |w_j| = -w_j$). The graph G_{st} is a *closure graph* (a closure graph is a graph with a source, a sink, and with all finite capacity arcs adjacent only to either the source or the sink.) This construction is illustrated in Figure 22.

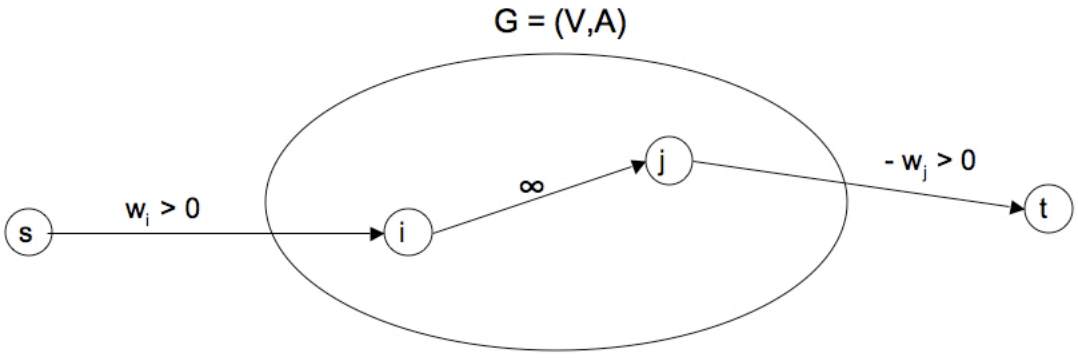


Figure 22: Visual representation of G_{st}.

Claim 9.6. *If $(s \cup S, t \cup T)$ is a finite $s-t$ cut on $G_{s,t}$, then S is a closed set on G.*

Proof. Assume by contradiction that S is not closed. This means that there must be an arc $(i,j) \in A$ such that $i \in S$ and $j \in T$. This arc must be on the cut (S,T), and by construction $u_{i,j} = \infty$, which is a contradiction on the cut being finite. □

Theorem 9.7. *If $(s \cup S, t \cup T)$ is an optimal solution to the minimum $s-t$ cut problem on $G_{s,t}$, then S is a maximum closed set on G.*

Proof.

$$\begin{aligned} C(s \cup S, t \cup T) &= \sum_{(s,i) \in A_{st},\ i \in T} u_{s,i} + \sum_{(j,t) \in A_{st},\ j \in S} u_{j,t} \\ &= \sum_{i \in T \cap V^+} w_i + \sum_{j \in S \cap V^-} -w_j \\ &= \sum_{i \in V^+} w_i - \sum_{i \in S \cap V^+} w_i - \sum_{j \in S \cap V^-} w_j \\ &= W^+ - \sum_{i \in S} w_i \end{aligned}$$

(Where $W^+ = \sum_{i \in V^+} w_i$, which is a constant.) This implies that minimizing $C(s \cup S, t \cup T)$ is equivalent to minimizing $W^+ - \sum_{i \in S} w_i$, which is in turn equivalent to

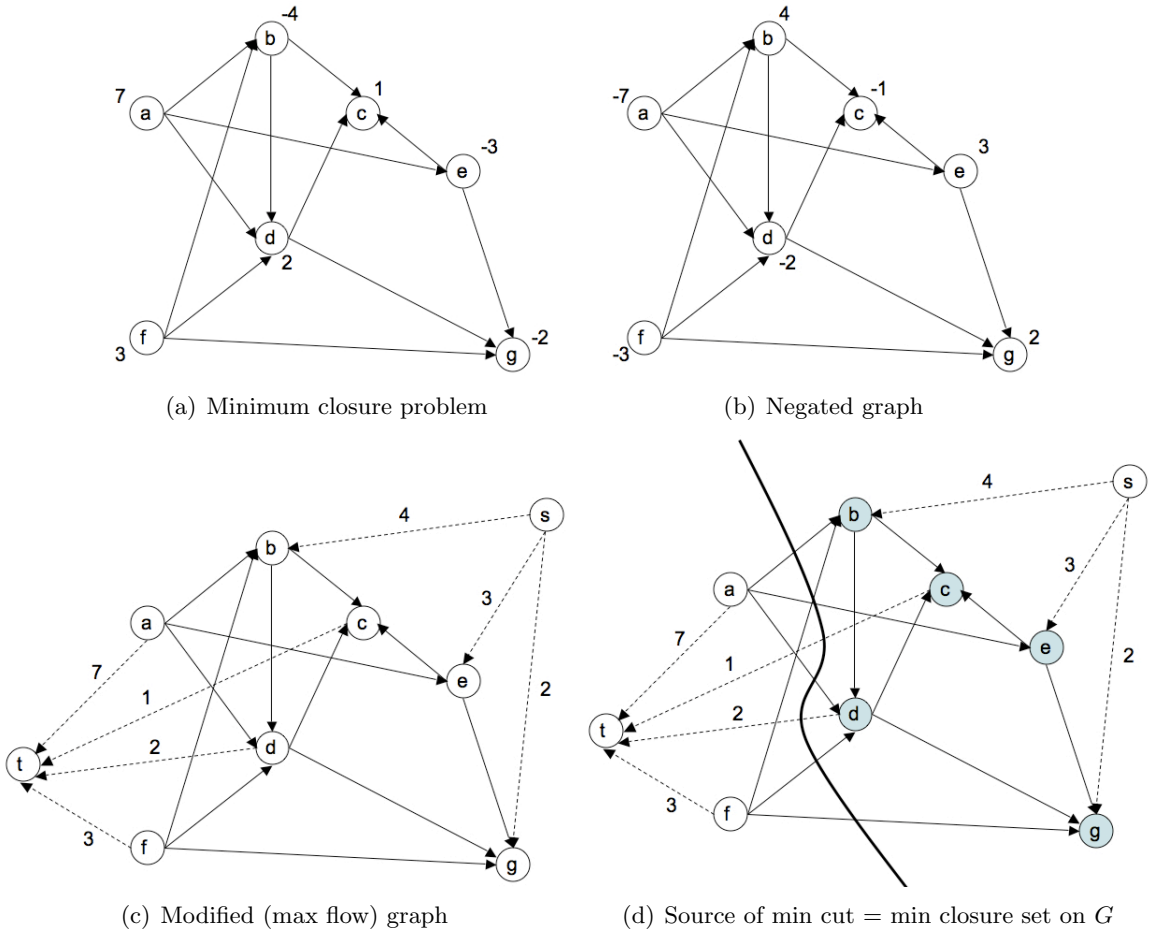
$$\max_{S \subseteq V} \sum_{i \in S} w_i$$

Therefore, any source set S that minimizes the cut capacity also maximizes the sum of the weights of the nodes in S. Since by Claim 9.6 any source set of an $s-t$ cut in $G_{s,t}$ is closed, we conclude that S is a maximum closed set on G. □

Variants / Special Cases

- In the minimum closure problem we seek to find a closed set with minimum total weight. This can be solved by negating the weights on the nodes in G to obtain G^-, constructing $G^-_{s,t}$ just as before, and solving for the maximum closure. Under this construction, the source set of a minimum $s-t$ cut on $G_{s,t}$ is a minimum closed set on G. See Figure 23 for a numerical example.
- The selection problem is a special case of the maximum closure problem, in that it is defined on a bipartite graph, rather than a general graph.

We comment that the maximum closure problem can also be presented as a selection problem on a bipartite graph. The bipartition consists of V^+ and V^-. There are arcs from a node in V^+ to all nodes in V^- that are reachable from it in the closure graph. Any closed set on this bipartite graph is closed in the original closure graph.



(a) Minimum closure problem

(b) Negated graph

(c) Modified (max flow) graph

(d) Source of min cut = min closure set on G

Figure 23: Converting a minimum closure problem to a maximum closure problem. Under this transformation, the source set of a minimum cut is also a minimum closed set on the original graph.

Numerical Example

9.3.4 The open-pit mining problem

Open-pit mining is a surface mining operation in which blocks of earth are extracted from the surface to retrieve the ore contained in them. During the mining process, the surface of the land is being continuously excavated, and a deeper and deeper pit is formed until the operation terminates. The final contour of this pit mine is determined before mining operation begins. To design the optimal pit – one that maximizes profit, the entire area is divided into blocks, and the value of the ore in each block is estimated by using geological information obtained from drill cores. Each block has a weight associated with it, representing the value of the ore in it, minus the cost involved in removing the block. While trying to maximize the total weight of the blocks to be extracted, there are also contour constraints that have to be observed. These constraints specify the slope requirements of the pit and precedence constraints that prevent blocks from being mined before others on top of them. Subject to these constraints, the objective is to mine the most profitable set of blocks.

The problem can be represented on a directed graph $G = (V, A)$. Each block i corresponds to a node with a weight b_i representing the net value of the individual block. The net value is computed as the assessed value of the ore in that block, from which the cost of extracting that block alone is deducted. There is a directed arc from node i to node j if block i cannot be extracted before block j which is on a layer right above block i. This precedence relationship is determined by the engineering slope requirements. Suppose block i cannot be extracted before block j, and block j cannot be extracted before block k. By transitivity this implies that block i cannot be extracted before block k. We choose in this presentation not to include the arc from i to k in the graph and the existence of a directed path from i to k implies the precedence relation. Including only arcs between immediate predecessors reduces the total number of arcs in the graph. Thus to decide which blocks to extract in order to maximize profit is equivalent to finding a maximum weight set of nodes in the graph such that all successors of all nodes are included in the set. This can be solved as the maximum closure problem in G described previously.

9.3.5 Forest clearing

The Forest Clearing problem is a real, practical problem. We are given a forest which needs to be selectively cut. The forest is divided into squares where every square has different kind of trees. Therefore, the value (or benefit from cutting) is different for each such square. The goal is to find the most beneficial way of cutting the forest while preserving the regulatory constraints.

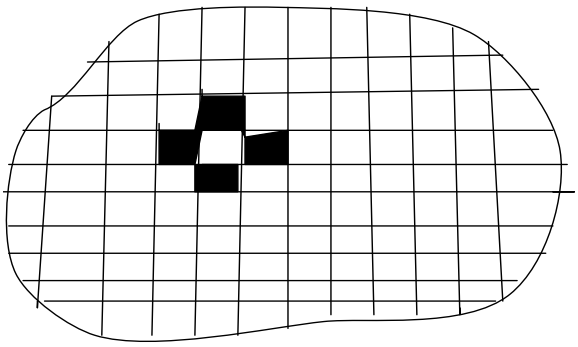
Version 1. One of the constraints in forest clearing is preservation of deer habitat. For example, deer like to eat in big open areas, but they like to have trees around the clearing so they can run for cover if a predator comes along. If they don't feel protected, they won't go into the clearing to eat. So, the constraint in this version of Forest Clearing is the following: no two adjacent squares can be cut. Adjacent squares are two squares which share the same side, i.e. adjacent either vertically or horizontally.

Solution. Make a *grid-graph* $G = (V, E)$. For every square of forest make a node v. Assign every node v, weight w, which is equal to the amount of benefit you get when cut the corresponding square. Connect two nodes if their corresponding squares are adjacent. The resulting graph is bipartite. We can color it in a chess-board fashion in two colors, such that no two adjacent nodes are of the same color. Now, the above Forest Clearing problem is equivalent to finding a max-weight independent set in the grid-graph G. In a previous section it was shown that Maximum Independent Set Problem is equivalent to Minimum Vertex Cover Problem. Therefore, we can solve our problem by solving weighted vertex cover problem on G by finding Min-Cut in the corresponding network.

This problem is solved in polynomial time.

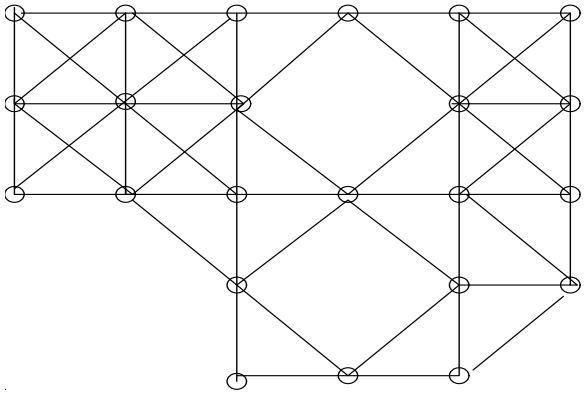
Version 2. Suppose, deer can see if the diagonal squares are cut. Then the new constraint is the following: no two adjacent vertically, horizontally or diagonally squares can be cut. We can build a grid-graph in a similar fashion as in *Version 1.* However, it will not be bipartite anymore, since it will have odd cycles. So, the above solution would not be applicable in this case. Moreover, the problem of finding Max Weight Independent Set on such a grid-graph with diagonals is proven to be *NP*-complete.

Another variation of Forest Clearing Problem is when the forest itself is not of a rectangular shape. Also, it might have "holes" in it, such as lakes. In this case the problem is also *NP*-complete.



The checkered board pattern permitted to clear

Because there are no odd cycles in the above grid graph of version 1, it is bipartite and the aforementioned approach works. Nevertheless, notice that this method breaks down for graphs where cells are neighbors also if they are adjacent diagonally. An example of such a graph is given below.



Such graphs are no longer bipartite. In this case, the problem has indeed been proven *NP*-complete in the context of producing memory chips (next application).

9.3.6 Producing memory chips (VLSI layout)

We are given a silicon wafer of 2G RAM VLSI chips arranged in a grid-like fashion. Some chips are damaged, those will be marked by a little dot on the top and cannot be used. The objective is to make as many 8G chips as possible out of good 2G chips. A valid 8G chip is the one where 4 little chips of 2G form a square, (i.e. they have to be adjacent to each other in a square fashion.) Create a grid-graph $G = (V, E)$ in the following manner: place a node in the middle of every possible

8G chip. (The adjacent 8G chips will overlap.) Similar to *Version 2* of Forest Clearing Problem put an edge between every two adjacent vertically, horizontally or diagonally nodes. Now, to solve the problem we need to find a maximum independent set on G. This problem is proven to be *NP*-complete.

9.4 Flow Decomposition

In many situations we would like, given a feasible flow, to know how to *recreate* this flow with a bunch of $s-t$ paths. In general a feasible flow might also contain cycles. This motivates the following definition.

A *primitive element* is either:

1. A simple path P from s to t with flow δ.
2. A cycle Γ with flow δ.

Theorem 9.8 (Flow Decomposition Theorem)**.** *Any feasible flow f can be decomposed into no more than m primitive elements.*

Proof. The proof is by construction; that is, we will give an algorithm to find the primitive elements. Let $A_f = \{(i,j) \in E : f_{ij} > 0\}$, and $G' = (V, A_f)$.

Pseudocode

While there is an arc in A_f adjacent to s (i.e. while there is a positive flow from s to t), do:
  Begin DFS in G_f from s, and stop when we either reach t along a simple path P, or found a simple cycle Γ. Let ξ denote the set of arcs of the primitive element found.
    Set $\delta = \min_{(i,j)\in\xi}\{f_{ij}\}$.
    Record either (P, δ) or (Γ, δ) as a primitive element.
    Update the flow in G by setting: $f_{ij} \leftarrow f_{ij} - \delta \ \forall (i,j) \in \xi$.
    Update A_f. Note that the number of arcs in A_f is reduced by at least one arc.

If no arc in A_f leaves s, then A_f must consist of cycles. While $A_f \neq \emptyset$ do:
  Pick any arc $(i,j) \in A_f$ and start DFS from i.
  When we come back to i (which is guaranteed by flow conservation), we found another primitive element, a cycle Γ.
    Set $\delta = \min_{(i,j)\in\Gamma}\{f_{ij}\}$.
    Record (Γ, δ) as a primitive element.
    Update the flow in G by setting: $f_{ij} \leftarrow f_{ij} - \delta \ \forall (i,j) \in \xi$.
    Update A_f. Note that the number of arcs in A_f

Note that the above algorithm is correct since, every time we update the flow, the new flow must still be a feasible flow.

Since, each time we found a primitive element we removed at least one arc from A_f, we can find at most m primitive elements. □

An interesting question is the complexity of the algorithm to generate the primitive elements and how it compares to maximum flow. We now show that the overall complexity is $O(mn)$. Assuming the we have for each node the adjacency list - the outgoing arcs. Follow a path from s in a DFS manner, visiting one arc at a time. If a node is repeated (visited twice) then we found a cycle. We trace it back and find its bottleneck capacity and record the primitive element. The bottleneck arcs are removed, and the adjacency lists affected are updated. This entire operation is done in $O(n)$ as we visit at most n nodes.

If a node is not repeated, then after at most n steps we reached the sink t. The path followed is then a primitive path with flow equal to its bottleneck capacity. We record it and update the flows and adjacency lists in $O(n)$ time.

Since there are $O(m)$ primitive elements, then all are found in $O(mn)$. Notice that this is (pretty much) faster than any known maximum flow algorithm.

9.5 Algorithms

First we introduce some terminology that will be useful in the presentation of the algorithms.
Residual graph: The residual graph, $G_f = (V, A_f)$, with respect to a flow f, has the following arcs:

- forward arcs: $(i,j) \in A_f$ if $(i,j) \in A$ and $f_{ij} < u_{ij}$. The residual capacity is $u^f_{ij} = u_{ij} - f_{ij}$. The residual capacity on the forward arc tells us how much we can increase flow on the original arc $(i,j) \in A$.
- reverse arcs: $(j,i) \in A_f$ if $(i,j) \in A$ and $f_{ij} > 0$. The residual capacity is $u^f_{ji} = f_{ij}$. The residual capacity on the reverse arc tells us how much we can decrease flow on the original arc $(i,j) \in A$.

Intuitively, residual graph tells us how much **additional** flow can be sent through the original graph with respect to the given flow. This brings us to the notion of an augmenting path.

In the presence of lower bounds, $\ell_{ij} \le f_{ij} \le u_{ij},\ (i,j) \in A$, the definition of forward arc remains, whereas for reverse arc, $(j,i) \in A_f$ if $(i,j) \in A$ and $f_{ij} > \ell_{ij}$. The residual capacity is $u^f_{ji} = f_{ij} - \ell_{ij}$.

Augmenting path: An augmenting path is a path from s to t in the residual graph. The *capacity of an augmenting path* is the minimum residual capacity of the arcs on the path – the bottleneck capacity.

If we can find an augmenting path with capacity δ in the residual graph, we can increase the flow in the original graph by adding δ units of flow on the arcs in the original graph which correspond to forward arcs in the augmenting path and subtracting δ units of flow on the arcs in the original graph which correspond to reverse arcs in the augmenting path.

Note that this operation does not violate the **capacity constraints** since δ is the smallest arc capacity in the augmenting path in the residual graph, which means that we can always add δ units of flow on the arcs in the original graph which correspond to forward arcs in the augmenting path and subtract δ units of flow on the arcs in the original graph which correspond to reverse arcs in the augmenting path without violating capacity constraints.

The **flow balance** constraints are not violated either, since for every node on the augmenting path in the original graph we either increment the flow by δ on one of the incoming arcs and increment the flow by δ on one of the outgoing arcs (this is the case when the incremental flow in the residual graph is along forward arcs) or we increment the flow by δ on one of the incoming arcs and decrease the flow by δ on some other incoming arc (this is the case when the incremental flow in the residual graph comes into the node through the forward arc and leaves the node through the reverse arc) or we decrease the flow on one of the incoming arcs and one of the outgoing arcs in the original graph (which corresponds to sending flow along the reverse arcs in the residual graph).

9.5.1 Ford-Fulkerson algorithm

The above discussion can be summarized in the following algorithm for finding the maximum flow on a give graph. This algorithm is is also known as the **augmenting path algorithm**. Below is

a pseudocode-like description.

Pseudocode:
f: flow;
G_f: the residual graph with respect to the flow;
P_{st}: a path from s to t;
u_{ij}: capacity of arc from i to j.

Initialize $f = 0$
If $\exists P_{st} \in G_f$ do
    find $\delta = \min_{(i,j)\in P_{st}} U_{ij}$
    $f_{ij} = f_{ij} + \delta \quad \forall (i,j) \in P_{st}$
else stop f is max flow.

Detailed description:

1. Start with a feasible flow (usually $f_{ij} = 0 \ \forall (i,j) \in A$).
2. Construct the residual graph G_f with respect to the flow.
3. Search for augmenting path by doing breadth-first-search from s (we consider nodes to be adjacent if there is a positive capacity arc between them in the residual graph) and seeing whether the set of s-reachable nodes (call it S) contains t.
   If S contains t then there is an augmenting path (since we get from s to t by going through a series of adjacent nodes), and we can then increment the flow along the augmenting path by the value of the smallest arc capacity of all the arcs on the augmenting path.
   We then update the residual graph (by setting the capacities of the forward arcs on the augmenting path to the difference between the current capacities of the forward arcs on the augmenting path and the value of the flow on the augmenting path and setting the capacities of the reverse arcs on the augmenting path to the sum of the current capacities and the value of the flow on the augmenting path) and go back to the beginning of step 3.
   If S does not contain t then the flow is maximum.

To establish correctness of the augmenting path algorithm we prove the following theorem which is actually stronger than the the max-flow min-cut theorem.

Reminder: f_{ij} is the flow from i to j, $|f| = \sum_{(s,i)\in A} f_{si} = \sum_{(i,t)\in A} f_{it} = \sum_{u\in S, v\in T} f_{uv}$ for any cut (S,T).

Theorem 9.9 (Augmenting Path Theorem)**.** *(generalization of the max-flow min-cut theorem) The following conditions are equivalent:*

1. *f is a maximum flow.*
2. *There is no augmenting path for f.*
3. *$|f| = U(S,T)$ for some cut (S,T).*

Proof.
$(1 \Rightarrow 2)$ If $\exists$ augmenting path p, then we can strictly increase the flow along p; this contradicts that the flow was maximum.

$(2 \Rightarrow 3)$ Let G_f be the residual graph w.r.t. f. Let S be a set of nodes reachable in G_f from s. Let $T = V \setminus S$. Since $s \in S$ and $t \in T$ then (S,T) is a cut. For $v \in S, w \in T$, we have the following

implications:

$$\begin{aligned}(v,w) \notin G_f &\Rightarrow f_{vw} = u_{vw} \text{ and } f_{wv} = 0 \\ &\Rightarrow |f| = \sum_{v\in S, w\in T} f_{vw} - \sum_{w\in T, v\in S} f_{wv} = \sum_{v\in S, w\in T} u_{vw} = U(S,T)\end{aligned}$$

$(3 \Rightarrow 1)$ Since $|f| \leq U(S,T)$ for any (S,T) cut, then $|f| = U(S,T) \implies f$ is a maximum flow. □

Note that the equivalence of conditions 1 and 3 gives the max-flow min-cut theorem.

Given the max flow (with all the flow values), a min cut can be found in by looking at the residual network. The set S, consists of s and the all nodes that s can reach in the final residual network and the set $\bar{S}$ consists of all the other nodes. Since s can't reach any of the nodes in $\bar{S}$, it follows that any arc going from a node in S to a node in $\bar{S}$ in the original network must be saturated which implies this is a minimum cut. This cut is known as the minimal source set minimum cut. Another way to find a minimum cut is to let $\bar{S}$ be t and all the nodes that can reach t in the final residual network. This a *maximal source set minimum cut* which is different from the minimal source set minimum cut when the minimum cut is not unique.

While finding a minimum cut given a maximum flow can be done in linear time, $O(m)$, we have yet to discover an efficient way of finding a maximum flow given a list of the edges in a minimum cut other than simply solving the maximum flow problem from scratch. Also, we have yet to discover a way of finding a minimum cut without first finding a maximum flow. Since the minimum cut problem is asking for less information than the maximum flow problem, it seems as if we should be able to solve the former problem more efficiently than the later one. More on this in Section 9.2.

In a previous lecture we already presented Ford-Fulkerson algorithm and proved its correctness. In this section we will analyze its complexity. For completeness purposes we give a sketch of the algorithm.

Ford-Fulkerson Algorithm
Step 0: $f = 0$
Step 1: Construct G_f (Residual graph with respect to flow f)
Step 2: Find an augmenting path from s to t in G_f
Let path capacity be δ
Augment f by δ along this path
Go to step 1
If there does not exist any path
Stop with f as a maximum flow.

Theorem 9.10. *If all capacities are integer and bounded by a finite number U, then the augmenting path algorithm finds a maximum flow in time $O(mnU)$, where $U = \max_{(v,w)\in A} u_{vw}$.*

Proof. Since the capacities are integer, the value of the flow goes up at each iteration by at least one unit.

Since the capacity of the cut $(s, N \setminus \{s\})$ is at most nU, the value of the maximum flow is at most nU.

From the two above observations it follows that there are at most $O(nU)$ iterations. Since each iteration takes $O(m)$ time–find a path and augment flow, then the total complexity is $O(nmU)$. □

The above result also applies for rational capacities, as we can scale to convert the capacities to integer values.

Drawbacks of Ford-Fulkerson algorithm:

1. The running time is not polynomial. The complexity is exponential (pseudo-polynomial) since it depends on U.
2. In Ford-Fulkerson's algorithm any augmenting path can be selected. In the graph of Figure 24, the algorithm could take 4000 iterations for a problem with maximum capacity of 2000. This is because the augmenting path is not "chosen wisely" (see Indiana Jones and the Last Crusade). We will see that a wise choice of the augmenting path leads to polynomial time algorithms.

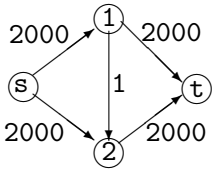


Figure 24: Graph leading to long running time for Ford-Fulkerson algorithm.

3. Finally, for irrational capacities, this algorithm may converge to the wrong value (see Papadimitriou and Steiglitz p.126-128)

Improvements for enhancing the performance of Ford-Fulkerson algorithm include:

1. Augment along maximum capacity augmenting paths.
2. Using the concept of capacity scaling to cope with the non-polynomial running time.
3. Augmenting along the shortest (in number of edges) path.

Each alternative leads to different types of max-flow algorithms discussed next.

9.5.2 Maximum capacity augmenting path algorithm

Dijkstra's algorithms is appropriate for use here, where the label of a node is the max capacity of a path from s. At each iteration we add to the set of permanent nodes a node with *largest* label reachable from the permanent set.

Algorithm

Initialize: $P \leftarrow \{s\}$ and $T \leftarrow \{V - s\}$

While T is not empty do

choose maximum label $l_i^* \in T$ and add it to the set of permanent nodes P.

update nodes in T.

$$l_j \in T \leftarrow \min_{(i,j)\in T}\{l_i^*, c_{ij}\}.$$

End Do

The complexity of finding the max capacity augmenting path is $O(m + n \log n)$.

The complexity of the max capacity augmenting path algorithm

Suppose that the value of the maximum flow, f^*, is v^*. We are at an iteration with flow f of value v. What is the maximum flow value on G_f? From the flow decomposition theorem, there are $\leq m$

paths from s to t with sum of flows equal to $v^* - v$. $\sum_{i=1,k} \delta_i = v^* - v$
$\Rightarrow \exists i | \delta_i \geq \frac{(v^*-v)}{k} \geq \frac{(v^*-v)}{m}$
$\Rightarrow$ remaining flow (after augmenting along a maximum capacity path) is $\leq \frac{m-1}{m}(v^* - v)$.
Repeat the above for q iterations, where $q = \frac{log(v^*)}{\log(\frac{m}{m-1})}$.

Stop when $(\frac{m-1}{m})^q v^* \leq 1$, since flows must be integers. Thus it suffices to have $q \geq \frac{log(v^*)}{\log(\frac{m}{m-1})}$. Now $\log(\frac{m}{m-1}) \approx \frac{1}{m}$. Thus the overall complexity is $O(m(m + n \log n) \log(nU))$.

Why is the largest capacity augmenting path not necessarily a primitive path? A primitive path never travels backwards along an arc, yet an augmenting path may contain backward arcs. Thus, knowing the flow in advance is a significant advantage (not surprisingly).

9.5.3 Capacity scaling algorithm

We note that the Ford-Fulkerson algorithm is good when our capacities are small. Most algorithms with this type of properties can be transformed to polynomial time algorithms using a *scaling* strategy.

The idea in this algorithm is to construct a series of max-flow problems such that: the number of augmentations on each problem is small; the maximum flow on the previous problem to find a feasible flow to the next problem with a small amount of work; the last problem gives the solution to our original max-flow problem.

In particular, at each iteration k, we consider the network P_k with capacities restricted to the k most significant digits (in the binary representation). (We need $p = \lfloor \log U \rfloor + 1$ digits to represent the capacities.) Thus, at iteration k, we will consider the network where the capacities are given by the following equation:

$$u_{ij}^{(k)} = \left\lfloor \frac{u_{ij}}{2^{p-k}} \right\rfloor$$

Alternatively, note that the capacities of the kth iteration can be obtained as follows:

$$u_{ij}^{(k)} = 2u_{ij}^{(k-1)} + \text{next significant digit}$$

From the above equation it is clear that we can find a feasible flow to the current iteration by doubling the max-flow from the previous iteration.

Finally, note that the residual flow in the current iteration can't be more than m. This is true since in the previous iteration we had a max-flow and a corresponding min-cut. The residual capacity at each arc of the min-cut from the previous scaling iteration can grow at most by one unit as these arcs were saturated in that previous iteration. Therefore the max-flow in the residual graph of the current iteration can be at most m units.

The preceding discussion is summarized in the following algorithm.

Capacity scaling algorithm
$f_{ij} := 0$;
Consider P_0 and apply the augmenting path algorithm.
For $k := 1$ to n Do
    Multiply all residual capacities and flows of residual graph from previous iteration by 2;
    Add 1 to all capacities of arcs that have 1 in the (k+1)st position
        (of the binary representation of their original capacity);
    Apply the augmenting path algorithm starting with the current flow;
End Do

Theorem 9.11. *The capacity scaling algorithm runs in $O(m^2 \log_2 U)$ time.*

Proof. For arcs on the cut in the previous network residual capacities were increased by at most one each, then the amount of residual max flow in P_i is bounded by the number of arcs in the cut which is $\leq m$. So the number of augmentations at each iteration is $O(m)$.
The complexity of finding an augmenting path is $O(m)$.
The total number of iterations is $O(\log_2 U)$.
Thus the total complexity is $O(m^2 \log_2 U)$. □

Notice that this algorithm is polynomial, but still not strongly polynomial. Also it cannot handle real capacities.

We illustrate the capacity scaling algorithm in Figure 25.

9.5.4 Dinic's algorithm for maximum flow

Dinic's algorithm is also known as the Blocking Flow Algorithm, or shortest augmenting paths algorithm. In order to describe this algorithm we need the following definitions.

Layered Network: Given a flow, f, and the corresponding residual graph, $G_f = (V, A_f)$, a *layered network* $AN(f)$ (AN stands for Auxiliary Network) is constructed as follows:
Using breadth-first-search in G_f, we construct a tree a rooted at s. Each node in V gets a distance label which is its distance in the BFS tree from s. Thus, the nodes that have an incoming arc from s get assigned a label of 1 and nodes at the next level of the tree get a label of 2, etc. Nodes with the same label are said to be at the same layer. Let the distance label of t be $k = d(t)$ then all nodes in layer $\geq k$ are removed from the layered network and all arcs are removed except those which go from a node at one layer to a node at the next consecutive layer.
Note that this layered network comprises all of the shortest paths from s to t in G_f.

Blocking flow: A flow is called blocking flow for network G' if every s, t path intersects with at least one saturated arc. Note that a blocking flow is not necessarily a maximum flow!

Stage k: We call stage k the augmentations along paths in the layered network where $k = d(t)$ (a network that includes k layers).

Now we are ready to give the basic implementation of Dinic's algorithm.

Dinic's algorithm

$f_{ij} = 0 \ \forall (i,j) \in A$
Use BFS to construct the first layered network $AN(f)$.
While $d(t) < \infty$ do:
    Find a blocking flow f' in $AN(f)$.
    Construct G_f and its respective $AN(f)$.
End While.

Find a blocking flow (naive implementation)

While DFS finds a path P from s to t do:
    Augment the flow along P.
    Remove saturated arcs in $AN(f)$.
    Cross out nodes with zero indegree or outdegree equal to zero.
End While.

Theorem 9.12. *Dinic's algorithm solves correctly the max-flow problem.*

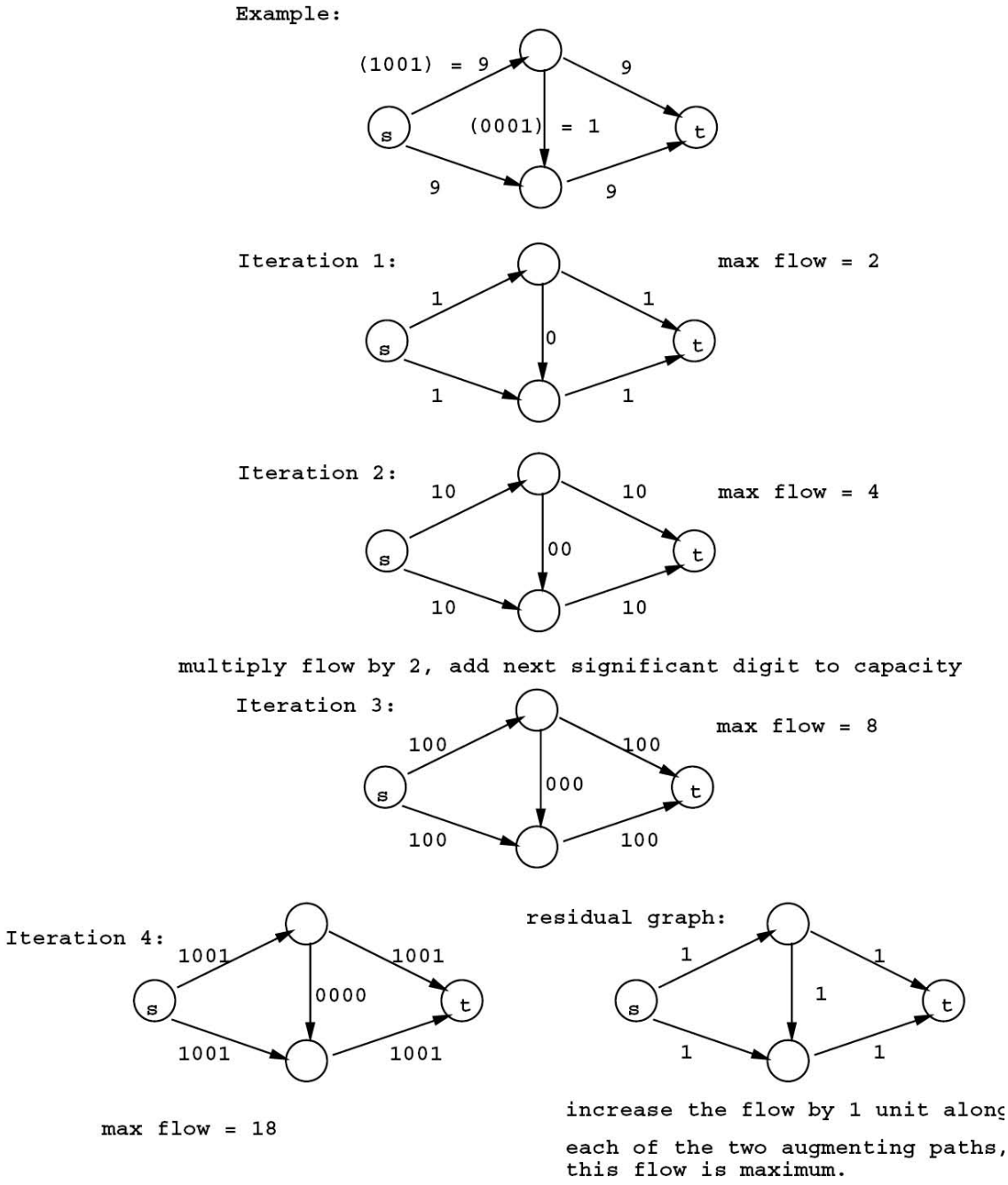


Figure 25: Example Illustrating the Capacity Scaling Algorithm

Proof. The algorithm finishes when s and t are disconnected in residual graph and so there is no augmenting path. Thus by the augmenting path theorem the flow found by Dinic's algorithm is a maximum flow. □

Complexity Analysis

We first argue that, after finding the blocking flow at the k-layered network, the shortest path in

the (new) residual graph is of length $\geq k+1$ (i.e. the level of t must strictly increase from one stage to the next one). This is true since all paths of length k intersect at least one saturated arc. Therefore, all the paths from s to t in the next layered network must have at least $k+1$ arcs (all paths with length $\leq k$ are already blocked). Thus, Dinic's algorithm has at most n stages (one stage for each layered network.

To find the blocking flow in stage k we do the following work: 1) Each DFS and update takes $O(k)$ time (Note that by definition of $AN(f)$ we can't have cycles.), and 2) we find the blocking flow in at most m updates (since each time we remove at least one arc from the layered network).

We conclude that Dinic's algorithm complexity is $O(\sum_1^n mk) = O(mn^2)$.

Now we will show that we can improve this complexity by finding more efficiently the blocking flow. In order to show this, we need to define one more concept.

Throughput: The *throughput* of a node is the minimum of the sum of incoming arcs capacities and the outgoing arcs capacity (that is, $thru(v) \equiv \min\left\{\sum_{(i,v)\in A^f} u^f_{v,j}, \sum_{(v,f)\in A^f} u^f_{v,f}\right\}$). The throughput of a node is the largest amount of flow that could possibly be routed through the node.

The idea upon which the improved procedure to find a blocking flow is based is the following. We find the node v with the minimum throughput $thru(v)$. We know that at this node we can "pull" $thru(v)$ units of flow from s, and "push" $thru(v)$ units of flow to t. After performing this pulling and pushing, we can remove at least one node from the layered network.

Find a blocking flow (improved)

While there is a node $v \neq s,t$ with $thru(v) > 0$ in $AN(f)$.
  Let v be the node in $AN(f)$ for which $thru(v)$ is the smallest; Set $\delta \leftarrow thru(v)$.
  Push δ from v to t.
  Pull δ from s to v.
  Delete (from $AN(f)$) all saturated arcs, v, and all arcs incident to v;
End While

Push δ from v to t

$Q \leftarrow \{v\}$; $excess(v) \leftarrow \delta$
While $Q \neq \emptyset$
  $i \leftarrow$ first node in Q (remove i from Q)
  For each arc (i,j), and while $excess(i) > 0$
    $\Delta \leftarrow \min\left\{u^f_{ij}, excess(i)\right\}$
    $excess(i) \leftarrow excess(i) - \Delta$
    $excess(j) \leftarrow excess(j) + \Delta$
    $thru(j) \leftarrow thru(j) - \Delta$
    $f_{ij} \leftarrow f_{ij} + \Delta$
    if $j \notin Q$ add j to end of Q
  End For
End While

The "Pull" procedure has a similar implementation.

Theorem 9.13. *Dinic's algorithm solves the max-flow problem for a network $G = (V, A)$ in $O(n^3)$ arithmetic operations.*

Proof. As before, the algorithm has $O(n)$ stages.

At each stage we need to create the layered network with BFS. This takes $O(m)$ complexity.

The operations at each stage can be either saturating push along an arc or an unsaturating push along an arc. So we can write number of operations as sum of saturating and unsaturating pull and push steps, $N = N_s + N_u$. We first note that once an arc is saturated, it is deleted from graph, therefore $N_s = O(m)$. However we may have many unsaturating steps for the same arc.

The key observation is that we have at most n executions of the push and pull procedures (along a path) at each stage. This is so because each such execution results in the deletion of the node v with the lowest throughput where this execution started. Furthermore, in each execution of the push and pull procedures, we have at most n unsaturated pushes – one for each node along the path. Thus, we have at each stage at most $O(n^2)$ unsaturated pushes and $N = N_s + N_u = O(m) + O(n^2) = O(n^2)$. Therefore, the complexity per stage is $O(m + n^2) = O(n^2)$.

We conclude that the total complexity is thus $O(n^3)$. □

Dinic's algorithm for unit capacity networks

Definition 9.14. *A unit capacity network is an $s - t$ network, where all arc capacities are equal to* 1*.*

Some problems that can be solved by finding a maximum flow in unit capacity networks are the following.

- Maximum Bipartite matching,
- Maximum number of edge disjoint Paths.

Lemma 9.15. *In a unit capacity network with distance from s to t greater or equal to ℓ the maximum flow value $|\hat{f}|$ satisfies $\sqrt{|\hat{f}|} \leq 2\frac{|V|}{\ell}$.*

Proof. Construct a layered network with V_i the set of the nodes in Layer i (See Figure 26). And let $S_i = V_0 \cup V_1 \cup \cdots \cup V_i$
$1 \leq i \leq \ell - 1$.

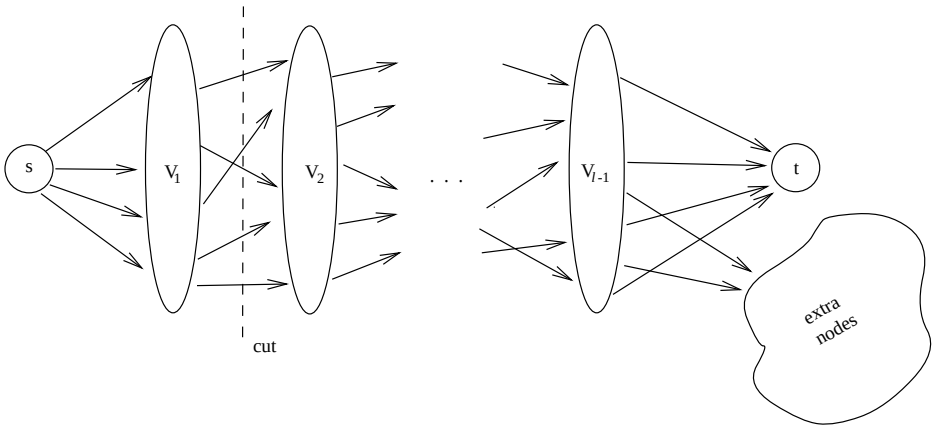


Figure 26: The layered network, distance from s to t is greater or equal to ℓ

Then $(S_i, \bar{S}_i)$ is a s-t cut. Using the fact that the maximum flow value is less than the capacity of any cut,

$$|\hat{f}| \leq |V_i||V_{i+1}| \Rightarrow \text{ either } |V_i| \geq \sqrt{|\hat{f}|} \text{ or } |V_{i+1}| \geq \sqrt{\hat{f}}$$

The first inequality comes from the fact that the maximum possible capacity of a cut is equal to $|V_i||V_{i+1}|$. Since each arc has a capacity of one, and the maximum of number of arcs from V_i toV_{i+1} is equal to $|V_i||V_{i+1}|$ (all the combinations of the set of nodes V_i and V_{i+1}).

So at least $\frac{\ell}{2}$ of the layers have at least $\sqrt{|\hat{f}|}$ nodes each
(consider the pairs, $V_0V_1, \ \ldots, \ V_2V_3, \ \ldots$).

$$\frac{\ell}{2}\sqrt{|\hat{f}|} \leq \sum_{i=1}^{\ell} |V_i| \leq |V| \Rightarrow \sqrt{|\hat{f}|} \leq \frac{2|V|}{\ell}$$

□

Claim 9.16. *Dinic's Algorithm, applied to the unit capacity network requires at most $O(n^{2/3})$ stages.*

Proof. Notice that in this case, it is impossible to have non-saturated arc processing, because of unit capacity, so the total computation per stage is no more than $O(m)$. If Max flow $\leq 2n^{2/3}$ then done. (Each stage increments the flow by at least one unit.)

If Max flow $> 2n^{2/3}$, run the algorithm for $2n^{2/3}$ stages. The distance labels from s to t will be increased by at least 1 for each stage. Let the Max flow in the residual network (after $2n^{2/3}$ stages) be g. The shortest path from s to t in this residual network satisfies $\ell \geq 2n^{2/3}$.

Apply lemma 9.15 to this residual network.

$$\sqrt{|g|} \leq \frac{2|V|}{2n^{2/3}} = n^{1/3}, |V| = n \Rightarrow |g| \leq n^{2/3}$$

It follows that no more than $n^{2/3}$ additional stages are required. Total number of stages $\leq 3n^{2/3}$. □

Complexity Analysis
Any arc processed is saturated (since this is a unit capacity network). Hence $O(m)$ work per stage. There are $O(n^{2/3})$ stages. This yields an overall complexity $O(mn^{2/3})$.
Remark: A. Goldberg and S. Rao [GR98] devised a new algorithm for maximum flow in general networks that use ideas of the unit capacity as subroutine. The overall running time of their algorithm is $O(\min\{n^{2/3}, m^{1/2}\}m \log(n^2/m) \log U)$ for U the largest capacity in the network.

Dinic's Algorithm for Simple Networks
In a simple network each node has either exactly one incoming arc of capacity one or exactly one outgoing arc of capacity one (See Figure 27 b). Bipartite matching is an example of a problem that can be formulated as a maximum flow problem on a simple network.

For simple networks, Dinic's Algorithm works more efficiently than it does for unit capacity networks, since the throughput of each node is one.

Lemma 9.17. *In a simple network, with distance from s to t greater than or equal to ℓ, the max flow, $\hat{f}$ satisfies $|\hat{f}| \leq \frac{|V|}{(\ell-1)}$*

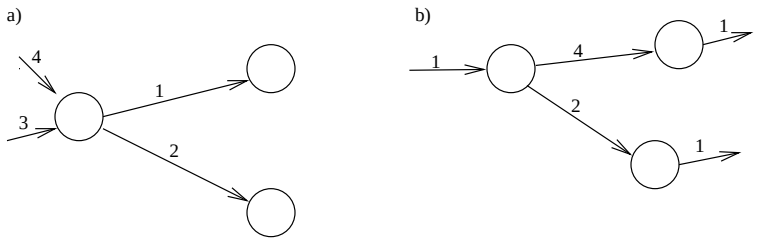


Figure 27: a) not a simple network b) a simple network

Proof. Consider layers $V_0, V_1 \ldots$ and the cut $(S_i, \bar{S}_i)$, where, $S_i = V_0 \cup \cdots \cup V_i$. Since for $i = 1 \ldots \ell - 1$ the throughput of each node for each layer V_i is one,

$$|\hat{f}| \leq |V_i| \text{ i} = 1 \ldots \ell\text{-}1$$

$$\text{Hence, } |\hat{f}| \leq \min_{i=1\ldots\ell-1} |V_i|$$

$$\sum_{i=1}^{\ell-1} |V_i| \leq |V| \rightarrow \min_{i=1\ldots.\ell-1} |V_i| \leq \frac{|V|}{(\ell-1)}$$

□

Claim 9.18. *Applying Dinic's Algorithm to a simple network, the number of stages required is* $O(\sqrt{n})$.

Proof. If the max flow $\leq \sqrt{n}$ done. Otherwise, run Dinic's algorithm for $\sqrt{n}$ stages. $l \geq \sqrt{n} + 1$ in that residual network, since each stage increments the flow by at least one unit and increases the distance label from the source to the sink by at least one at each stage. In the residual network, the Maxflow g satisfies $|g| \leq \frac{|V|}{(\sqrt{n}+1)} \leq \sqrt{n}$. So, we need at most $\sqrt{n}$ additional stages. Thus the total number of the stages is $O(\sqrt{n})$ □

Complexity Analysis

Recall we are considering of simple networks. All push/pull operations are saturating a node since there is only one incoming arc or outgoing arc with capacity 1 each node v. The processing thus makes the throughput of node v be 0.

Note that even though we only have to check $O(n)$ nodes, we still have to update the layered network which requires $O(m)$ work. Hence the work per stage is $O(m)$.

Thus, the complexity of the algorithm is $O(m\sqrt{n})$

References

[Din1] E. A. Dinic. Algorithm for Solution of a Problem of Maximum Flows in Networks with Power Estimation. *Soviet Math. Dokl.*, 11, 1277-1280, 1970.

[EK1] J. Edmonds and R. M. Karp. Theoretical Improvements in Algorithmic Efficiency for Network Flow Problems. *J. of ACM*, 19, 248-264, 1972.

[FF1] L. R. Ford, Jr. and D. R. Fulkerson. Maximal Flow through a Network. *Canad. J. Math.*, 8, 399-404, 1956.

[GR98] A. V. Goldberg and S. Rao. Beyond the flow decomposition barrier. *Journal of the ACM* 45, 783–797, 1998.

[GT1] A. V. Goldberg and R. E. Tarjan. A new approach to the maximum flow problem. *J. of ACM*, 35, 921-940, 1988.

[Pic1] J. C. Picard, "Maximal Closure of a Graph and Applications to Combinatorial Problems," *Management Science*, 22 (1976), 1268-1272.

[Jo1] T. B. Johnson, Optimum Open Pit Mine Production Technology, *Operations Research Center*, University of California, Berkeley, Ph.D. Dissertation., 1968.

10 Goldberg's Algorithm - the Push/Relabel Algorithm

10.1 Overview

All algorithms discuss till now were "primal" algorithms, that is, they were all based on augmenting feasible flows. The pus/relabel algorithm for the maximum flow problem is basd on using "preflows". This algorithm was originally developed by Goldberg and is known as the *Preflow Algorithm*. A generic algorithm and a particular variant (*lift-to-front*) will be presented.

Goldberg's algorithm can be contrasted with the algorithms we have learned so far in following ways:

- As opposed to working on feasible flows at each iteration of the algorithm, preflow algorithm works on preflows: the flows through the arcs need not satisfy the flow balance constraints.
- There is never an augmenting path in the residual network. The preflow is possibly infeasible, but super-optimal.

Definitions

Preflow: Preflow $\vec{f}$ in a network is defined as the one which

1. Satisfies the capacity constraints (i.e., $0 \leq f_{ij} \leq U_{ij}, \ \ \forall (i,j)$).
2. The incoming flow at any node except the source node must be greater than or equal to the outgoing flow, i.e.,
$$\Sigma_j f_{ji} - \Sigma_k f_{ik} \geq 0$$

Excess: Excess of a node is defined w.r.t. a flow and is given by

$$e_f(v) = \text{ incoming flow}(v) - \text{ outgoing flow}(v).$$

Active Node: A node $v \neq t$ is called *active* if $e_f(v) > 0$.

Distance Labels: Each node v is assigned a label. The labeling is such that it satisfies the following inequality
$$d(i) \leq d(j) + 1 \ \ \forall (i,j) \in A_f.$$

Admissible Arc An arc (u,v) is called *admissible* if $(u,v) \in A_f$ and $d(u) > d(v)$. Notice that together with the validity of the distance labels this implies that $d(u) = d(v) + 1$.

Lemma 10.1. *The admissible arcs form a directed acyclic graph (DAG).*

Proof: For contradiction suppose the cycle is given as $(i_1, i_2, ..., i_k, i_1)$. From the definition of an admissible arc, $d(i_1) > d(i_2) > \ldots > d(i_k) > d(i_1)$, from which the contradiction follows. □

Lemma 10.2. *The distance label of node i, $d(i)$, is a lower bound on the shortest path distance in A_f from i to t, if such path exists.*

Proof: Let the shortest path from i to t consist of k arcs: $(i, i_1, ..., i_{k-1}, t)$. From the validity of the distance labels,

$$d(i) \leq d(i_1) + 1 \leq d(i_2) + 2 \leq \ldots \leq d(t) + k = k.$$

□

Corollary 10.3. *If the distance label of a node i, satisfies $d(i) \geq n$, then there is no path in A_f from i to t. That is, t is not reachable from i.*

10.2 The Generic Algorithm

```
begin
    /* Preprocessing steps */
    f = 0
    d(s) = n, d(i) = 0,  ∀i ≠ s
    ∀(s, j) ∈ E  f_sj = u_sj
    e_f(j) = u_sj
    while network contains an active node do
                begin
                    /* Push step */
                    select an active node (v)
                    while ∃ admissible arcs (v, w)
                     then
                        Let δ = min{e_f(v), U_f(v, w)}, where (v, w) is admissible
                        Send δ units of flow from v to w.
                     else
                    /* Relabel step (all (v, w) are inadmissible) */
                    set d(v) = min_{(v,w)∈A_f} {d(w) + 1}
                end
end
```

Now we prove the following lemmas about this algorithm which will establish its correctness.

Lemma 10.4. *Relabeling step maintains the validity of distance labels.*

Proof: Relabeling is applied to node i with $d(i) \leq d(j)\ \ \forall (i, j) \in A_f$. Suppose the new label is

$$d'(i) = \min_{(i,j) \in A_f} \{d(j) + 1\}$$

This implies

$$d'(i) \leq d(j) + 1 \quad \forall (i, j) \in A_f$$

□

Lemma 10.5 (Super optimality)**.** *For preflow $\vec{f}$ and distance labeling $\vec{d}$ there is no augmenting path from s to t in the residual graph G_f.*

Proof: Since $d(s) = n$ this follows directly from Corollary 10.3. □

Lemma 10.6 (Optimality)**.** *When there is no active node in the residual graph* ($\forall v,\ e_v = 0$) *preflow is a maximum flow.*

Proof: Since there is no node with positive excess, the preflow is a feasible flow. Also from the previous lemma there is no augmenting path, hence the flow is optimal. □

Lemma 10.7. *For any active node i, there exists a path in G_f from i to s.*

Proof: Let S_i be the nodes reachable in G_f from i. If $s \notin S_i$, then all nodes in S_i have non-negative excess. (Note that s is the only node in the graph that is allowed to have a negative excess - deficit.) In-flow into S_i must 0 (else can reach more nodes than S_i)

$$\begin{aligned} 0 &= inflow(S_i) \\ &\geq inflow(S_i) - outflow(S_i) \\ &= \sum_{j \in S_i} [inflow(j) - outflow(j)] \\ &= \sum_{j \in S_i \setminus \{i\}} [inflow(j) - outflow(j)] + e_f(i) \\ &> 0. \end{aligned}$$

Since the node is active, hence it has strictly positive excess. Therefore, a contradiction. $\Rightarrow s \in S_i \Rightarrow$ there exists a path in G_f from i to s. □

Lemma 10.8. *When all active nodes satisfy $d(i) \geq n$, then the set of nodes $S = \{i | d(i) \geq n\}$ is a source set of a min-cut.*

Proof: Firstly, there is no residual (augmenting) path to t from any node in S (Corollary 10.3). Let $S^+ = \{i \mid d(i) \geq n,\ e(i) > 0\}$
From Lemma 10.7, each active node can send back excess to s. Hence, the preflow can be converted to (a feasible) flow and the saturated cut $(S, \bar{S})$ is a min-cut. Therefore the resulting feasible flow is maximum flow. (Although the conversion of preflow to a feasible flow can be accomplished by the algorithm, it is more efficient to send flow from excess nodes back to source by using the flow decomposition algorithm. That is, terminate the push-relabel algorithm when there is no active node of label $< n$.) □

Note that when we terminate the push-relabel algorithm when there is no active node of label $< n$, any node of label $\geq n$ must be active. This is since when it was last relabeled it was active, and could not have gotten rid of the access as of yet.

Notice also that the source set S is minimal source set of a min cut. The reason is that the same paths that are used to send the excess flows back to source, can then be used in the residual graph to reach those nodes of S.

Lemma 10.9. $\forall v,\ \ d(v) \leq 2n - 1$

Proof: Consider an active node v. From lemma 10.7, s is reachable from v in G_f. Consider a simple path, P, from v to s. $P = (i_0, i_1, ...i_k)$ where $i_0 = v, i_k = s$. Now,

$$\begin{aligned} d(s) &= n \text{ (fixed)} \\ d(i_0) &\leq d(i_1) + 1 \leq d(i_2) + 2..... \leq d(i_k) + k \\ d(i_0) &\leq d(i_k) + k = n + k \\ d(i_0) &\leq n + k \end{aligned}$$

Since any simple path has no more that n nodes, we have $k \leq n-1$, which implies that $\Rightarrow d(i_0) \leq 2n-1$. □

Lemma 10.10. *1. $d(v)$ never decreases when relabeled*

2. Immediately after relabel(v), v does not have incoming admissible arcs

Proof:

1. Before *relabel*(v), $d(v) \leq d(w)\ \ \forall (v,w) \in A_f$. After relabel the new label is $d(w)+1 \geq d(v)+1$ $\Rightarrow d(v)$ increased after relabeling.

2. Assume $(u,v) \in A_f$ and previously admissible $\Rightarrow d(u) = d(v)+1$. After relabel $\bar{d}(v)$ is the new label and $\bar{d}(v) \geq d(v)+1 \geq d(u)$. Since v is now at least as high as u, (u,v) is inadmissible.

□

Lemma 10.11 (Complexity of Relabel)**.**

1. Total number of relabels $= O(n^2)$

2. Complexity of all relabels $= O(mn)$

Proof:

1. Because of Lemma 10.9 each node can be relabeled at most 2n-1 times. Total number of nodes = n $\Rightarrow$ Total number of all relabels is less than $n(2n-1) = O(n^2)$

2. The cost of relabel(v) is equal to the out-degree of v in G_f because for each v, we have to check all its adjacent nodes. Therefore, the cost of relabeling each node throughout the algorithm is $\leq n \times deg(v)$. The cost of relabeling all nodes is then at most $\sum_{v \in V} n \times deg(v) = 2mn$.

□

Lemma 10.12 (Complexity of Push)**.** *Total number of saturating pushes is $O(mn)$*

Proof: After a saturating push on (u,v), we cannot push again to v, unless v pushes on (v,u). Therefore $d(v)$ has increased by at least 2 units and hence $d(u)$ must also have increased by at least 2 units between saturated pushes on (u,v) (See Figure 28.)
$\Rightarrow$ number of saturated pushes on $(u,v) \leq \frac{2n-1}{2} \leq n$
Therefore, for $O(m)$ arcs, total number of saturating pushes $= O(mn)$ □

Lemma 10.13. *Total number of non saturating pushes $= O(n^2m)$*

Proof: Define $\Phi = \sum_{v\ active} d(v)$
Initially $\Phi = 0$ because all initial labels are zero. A non-saturating push (v,w) reduces Φ by at least 1, because it deactivates node v and reduces Φ by $d(v)$. Note that $d(v) \geq 1$ always for a push(v,w) to occur. But d(w) may now become active and add to Φ. Therefore $\Phi' = \Phi + d(w) - d(v)$. However $d(v) > d(w) \Rightarrow \Phi$ is reduced by at least 1. In fact the decrease is exactly one if w was not previsouly active, since $d(v) \leq d(w)+1$ for valid labeling; $d(v) \geq d(w)+1$ for push(v,w) to occur. The total number of non saturating pushes is bounded by the total amount that Φ can increase during the algorithm. Φ can increase due to:

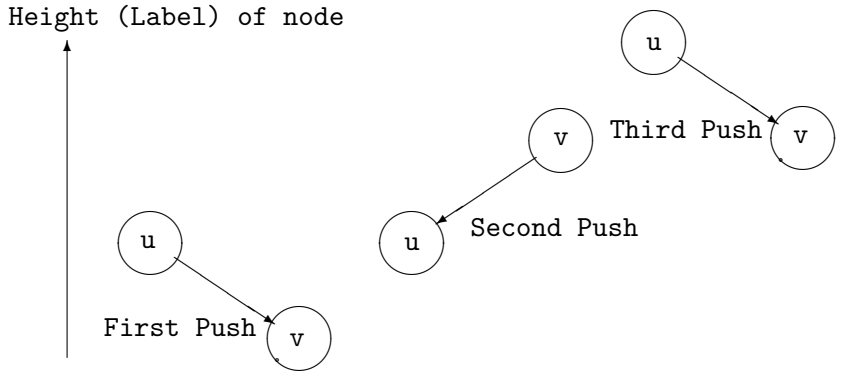


Figure 28: Sequence of saturating pushes on (u, v)

1. Saturating push :
   A saturating push can increase Φ by at most 2n-2 since it can activate at most one node and $d(v) \leq 2n - 1$.
   Total amount the Φ can increase per saturating push = $O(n)$
   Total number of saturating pushes = $O(nm)$
   Total amount that Φ can increase due to all saturating pushes = $O(n^2m)$.

2. Relabeling:
   Each relabel increases Φ by the increase in the label.

   Maximum increase of label per node = $2n - 1$
   Total increase in Φ by relabeling = $O(n^2)$. This term is dominated by $O(n^2m)$.

Since total increase = $O(n^2m)$, the number of non-saturating pushes is at most $O(n^2m)$ □

Overall Complexity

Cost of relabels = $O(nm)$
Cost of saturating pushes = $O(nm)$
Cost of non-saturating pushes = $O(n^2m)$
Therefore total complexity is $O(n^2m)$.

10.3 Variants of Push/Relabel Algorithm

- FIFO: A node that becomes active is placed at the beginning of the active node list. Running time = $O(n^3)$
- Highest label preflow: picks an active node to process that has the highest distance label $O(\sqrt{m}n^2)$
- Dynamic tree implementation: Running time = $O(mn \frac{log\, n^2}{m})$

10.4 Wave Implementation of Goldberg's Algorithm (Lift-to-front)

This algorithm has running time $O(n^3)$. The algorithm is listed below:

```
preprocess()
While (there exists an active node(i))
      Do
          if v is active then push/relabel v
          if v gets relabeled, place v in front of L
          else replace v by the node after v in L
      end
end
```

See the handout titled *Wave Algorithm.*
(from Cormen. Leiserson and Rivest).

Since the run time for relabel and saturating push are less than $O(n^3)$, we only have to show the run time of non-saturating push to be $O(n^3)$ to proof the overall running time to be $O(n^3)$.

Lemma 10.14. *The complexity of wave algorithm is* $O(n^3)$.

Proof. It suffices to show that there are $O(n^3)$ non-saturating pushes for the wave algorithm. Define a phase to be the period between two consecutive relabels.
There are $\leq n(2n-1)$ relabels. So there are $O(n^2)$ phases and we need to show that for each node there are ≤ 1 non-saturating push per phase. (Lemma 10.16.)

Claim 10.15. *The list* L *is in topological order for the admissible arcs of* A_f.

Proof. By induction on the iteration index. After preprocess, there are no admissible arcs and so any order is topological. Given a list in topological order, process node u. 'Push' can only eliminate admissible arcs (with a saturating push) and 'relabel' can create admissible arcs, but none into u (lemma 10.10), so moving u to the front of L maintains topological order.

Lemma 10.16. *At most one non-saturating push per node per phase.*

Proof. Let u be a node from which a non-saturating push is executed. Then u becomes inactive. Also all the nodes preceding u in L are inactive. In order for u to become active, one node preceding it has to become active. But every node can only make nodes lower than itself active. Hence the only way for u to become active is for some node to be relabeled.

It follows that the complexity of wave implementation of Goldberg's algorithm is $O(n^3)$.

11 The pseudoflow algorithm

In this section we provide a description of the pseudoflow algorithm. Our description here is different from than in [Hoc08] although the algorithm itself is the same. This description uses terminology and concepts similar in spirit to push-relabel in order to help clarify the similarities and differences between the two algorithms.

The first step in the pseudoflow algorithm, called the *Min-cut Stage* finds a minimum cut in G_{st}. Source-adjacent and sink-adjacent arcs are saturated throughout this stage of the algorithm; consequently, the source and sink have no role to play in the Min-cut Stage.

The algorithm may start with any other pseudoflow that saturates arcs in $A_s \cup A_t$. Other than that, the only requirement of this pseudoflow is that the collection of *free arcs*, namely the arcs that satisfy $\ell_{ij} < f_{ij} < u_{ij}$, form an acyclic graph.

Each node in $v \in V$ is associated with at most one *current arc* $(u, v) \in A^f$; the corresponding *current node* of v is denoted by current$(v) = u$. The set of current arcs in the graph satisfies the following invariants at the beginning of every major iteration of the algorithm:

Property 1. *(a) The graph does not contain a cycle of current arcs.*
(b) If $e(v) \neq 0$, then node v does not have a current arc.

Each node is associated with a *root* that is defined constructively as follows: starting with node v, generate the sequence of nodes $\{v, v_1, v_2, \ldots, v_r\}$ defined by the current arcs $(v_1, v), (v_2, v_1), \ldots, (v_r, v_{r-1})$ until v_r has no current arc. Such node v_r always exists, as otherwise a cycle will be formed, which would violate Property 1(a). Let the unique root of node v be denoted by root(v). Note that if a node v has no current arc, then root$(v) = v$.

The set of current arcs forms a *current forest*. Define a *component* of the forest to be the set of nodes that have the same root. It can be shown that each component is a directed tree, and the following properties hold for each component.

Property 2. *In each component of the current forest,*
(a) The root is the only node without a current arc.
(b) All current arcs are pointed away from the root.

While it is tempting to view the ability to maintain pseudoflows as an important difference between the two algorithms, it is trivial to modify the push-relabel algorithm (as shown in [CH09]) to handle pseudoflows.

The key differences between the pseudoflow and push-relabel algorithms are: (1) that the pseudoflow algorithm allows flow to be pushed along arcs (u, v) in which $\ell(u) = \ell(v)$ whereas this is not allowed in push-relabel; and (2) that flow is pushed along a path rather than along single arcs. Note that [GR98] proposed a maximum flow algorithm with complexity superior to that of push-relabel that relied on being able to send flow along arcs with $\ell(u) = \ell(v)$.

11.1 Initialization

The pseudoflow algorithm starts with a pseudoflow and an associated current forest, called in [Hoc08], a *normalized tree*. Each non-root in a branch (tree) of the forest has a parent node, and the unique arc from the parent node is called *current arc*. The root has no parent node and no current arc.

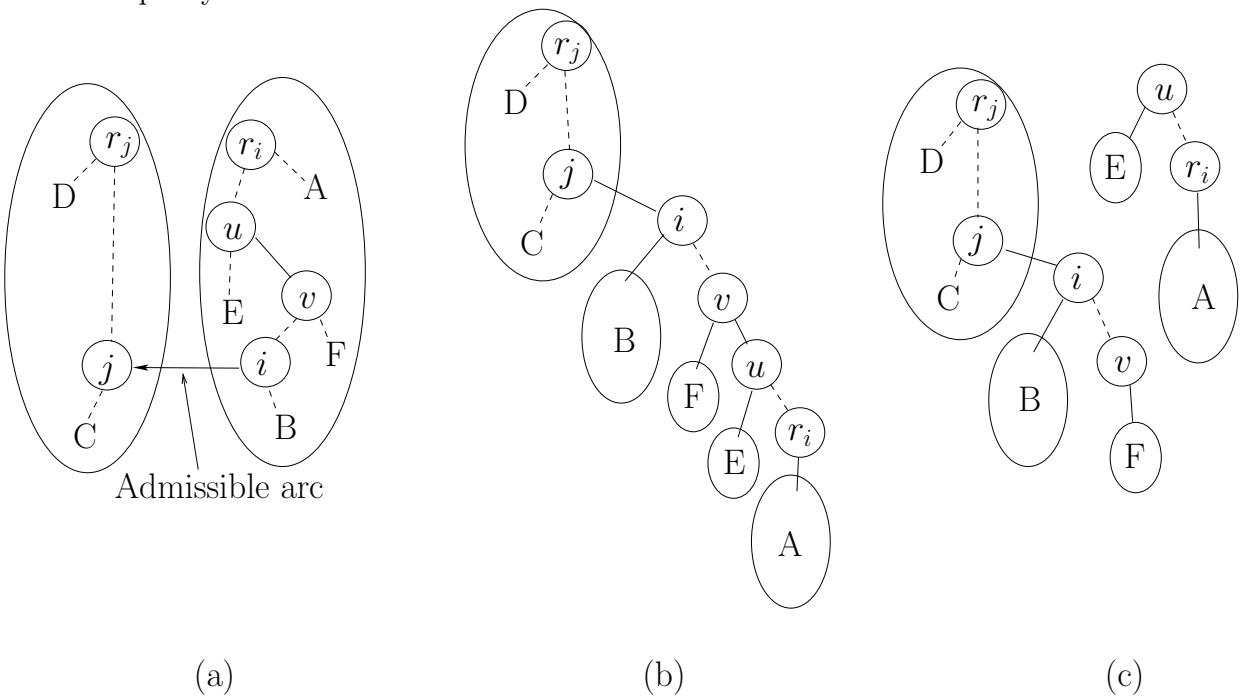
The generic initialization is the *simple* initialization: source-adjacent and sink-adjacent arcs are saturated while all other arcs have zero flow.

If a node v is both source-adjacent and sink-adjacent, then at least one of the arcs (s, v) or (v, t) can be pre-processed out of the graph by sending a flow of $\min\{c_{sv}, c_{vt}\}$ along the path $s \to v \to t$. This flow eliminates at least one of the arcs (s, v) and (v, t) in the residual graph. We henceforth assume w.l.o.g. that no node is both source-adjacent and sink-adjacent.

The simple initialization creates a set of source-adjacent nodes with excess, and a set of sink-adjacent nodes with deficit. All other arcs have zero flow, and the set of current arcs is selected to be empty. Thus, each node is a singleton component for which it serves as the root, even if it is *balanced* (with 0-deficit).

A second type of initialization is obtained by saturating all arcs in the graph. The process of saturating all arcs could create nodes with excesses or deficits. Again, the set of current arcs is

Figure 29: (a) Components before merger (b) Before pushing flow along admissible path from r_i to r_j (c) New components generated when arc (u, v) leaves the current forest due to insufficient residual capacity.



(a) (b) (c)

empty, and each node is a singleton component for which it serves as the root. We refer to this as the *saturate-all initialization* scheme.

11.2 A labeling pseudoflow algorithm

In the labeling pseudoflow algorithm, each node $v \in V$ is associated with a distance label $\ell(v)$ with the following property.

Property 3. *The node labels satisfy:*
(a) For every arc $(u, v) \in A^f$, $\ell(u) \leq \ell(v) + 1$.
(b) For every node $v \in V$ with strictly positive deficit, $\ell(v) = 0$.

Collectively, the above two properties imply that $\ell(v)$ is a lower bound on the distance (in terms of number of arcs) in the residual network of node v from a node with strict deficit. A residual arc (u, v) is said to be *admissible* if $\ell(u) = \ell(v) + 1$.

A node is said to be *active* if it has strictly positive excess. Given an admissible arc (u, v) with nodes u and v in different components, we define an *admissible path* to be the path from root(u) to root(v) along the set of current arcs from root(u) to u, the arc (u, v), and the set of current arcs (in the reverse direction) from v to root(v).

We say that a component of the current forest is a *label-n component* if for every node v of the component $\ell(v) = n$. We say that a component is a *good active component* if its root node is active and if it is not a label-n component.

An iteration of the pseudoflow algorithm consists of choosing a good active component and attempting to find an admissible arc from a *lowest labeled* node u in this component. (Choosing a lowest labeled node for processing ensures that an admissible arc is never between two nodes

of the same component.) If an admissible arc (u, v) is found, a *merger* operation is performed. The merger operation consists of pushing the entire excess of root(u) towards root(v) along the admissible path and updating the excesses and the arcs in the current forest to preserve Property 1.

If no admissible arc is found, $\ell(u)$ is increased by 1 unit. A schematic description of the merger operation is shown in Figure 29. The pseudocode for the generic labeling pseudoflow algorithm is given in Figures 30 through 32.

```
/*
Min-cut stage of the generic labeling pseudoflow algorithm. All nodes in label-n components form the nodes in the source set of the min-cut.
*/
procedure GenericPseudoflow (V_st, A_st, c):
    begin
        SimpleInit (A_s, A_t, c);
        while ∃ a good active component T do
            Find a lowest labeled node u ∈ T;
            if ∃ admissible arc (u, v) do
                Merger (root(u), ··· , u, v, ··· , root(v));
            else do
                ℓ(u) ← ℓ(u) + 1;
    end
```

Figure 30: Generic labeling pseudoflow algorithm.

```
/*
Saturates source- and sink-adjacent arcs.
*/
procedure SimpleInit(A_s, A_t, c):
    begin
        f, e ← 0;
        for each (s, i) ∈ A_s do
            e(i) ← e(i) + c_si; ℓ(i) = 1;
        for each (i, t) ∈ A_t do
            e(i) ← e(i) − c_it; ℓ(i) = 0;
        for each v ∈ V do
            ℓ(v) ← 0;
            current(v) ← ∅;
    end
```

Figure 31: Simple initialization in the generic labeling pseudoflow algorithm.

11.3 The monotone pseudoflow algorithm

In the generic labeling pseudoflow algorithm, finding the lowest labeled node within a component may take excessive time. The *monotone implementation* of the pseudoflow algorithm efficiently finds

```
/*
Pushes flow along an admissible path and preserves invariants.
*/
procedure Merger(v_1, ··· , v_k):
    begin
        for each j = 1 to k − 1 do
            if e(v_j) > 0 do
                δ ← min{c(v_j, v_{j+1}), e(v_j)};
                e(v_j) ← e(v_j) − δ;
                e(v_{j+1}) ← e(v_{j+1}) + δ;
            if e(v_j) > 0 do
                current(v_j) ← ∅;
            else do
                current(v_j) ← v_{j+1};
    end
```

Figure 32: Push operation in the generic labeling pseudoflow algorithm.

the lowest labeled node within a component by maintaining an additional property of *monotonicity* among labels in a component.

Property 4 ([Hoc08])**.** *For every current arc* (u, v)*,* $\ell(u) = \ell(v)$ *or* $\ell(u) = \ell(v) - 1$*.*

This property implies that within each component, the root is the lowest labeled node and node labels are non-decreasing with their distance from the root. Given this property, all the lowest labeled nodes within a component form a sub-tree rooted at the root of the component. Thus, once a good active component is identified, all the lowest labeled nodes within the component are examined for admissible arcs by performing a depth-first-search in the sub-tree starting at the root.

In the generic labeling algorithm, a node was relabeled if no admissible arc was found from the node. In the monotone implementation, a node u is relabeled only if no admissible arc is found *and* for all current arcs (u, v) in the component, $\ell(v) = \ell(u) + 1$. This feature, along with the merger process, inductively preserves the monotonicity property. The pseudocode for the Min-cut Stage of the monotone implementation of the pseudoflow algorithm is given in Figure 33.

The monotone implementation simply delays relabeling of a node until a later point in the algorithm, which does not affect correctness of the labeling pseudoflow algorithm.

11.4 Complexity summary

In the monotone pseudoflow implementation, the node labels in the admissible path are non-decreasing. To see that notice that for a merger along an admissible arc (u, v) the nodes along the path $\text{root}(u), \cdots, u$ all have equal label and the nodes along the path $v, \cdots, \text{root}(v)$ have non-decreasing labels (from Property 4). A merger along an admissible arc (u, v) either results in arc (u, v) becoming current, or in (u, v) leaving the residual network. In both cases, the only way (u, v) can become admissible again is for arc (v, u) to belong to an admissible path, which would require $\ell(v) \geq \ell(u)$, and then for node u to be relabeled at least once so that $\ell(u) = \ell(v) + 1$. Since $\ell(u)$ is bounded by n, (u, v) can lead to $O(n)$ mergers. Since there are $O(m)$ residual arcs, the number of mergers is $O(nm)$.

```
/*
Min-cut stage of the monotone implementation of pseudoflow algorithm. All nodes in
label-n components form the nodes in the source set of the min-cut.
*/
procedure MonotonePseudoflow (V_st, A_st, c):
    begin
        SimpleInit (A_s, A_t, c);
        while ∃ a good active component T with root r do
            u ← r;
            while u ≠ ∅ do
                if ∃ admissible arc (u, v) do
                    Merger (root(u), ··· , u, v, ··· , root(v));
                    u ← ∅;
                else do
                    if ∃w ∈ T : (current(w) = u) ∧ (ℓ(w) = ℓ(u)) do
                        u ← w;
                    else do
                        ℓ(u) ← ℓ(u) + 1;
                        u ← current(u);
    end
```

Figure 33: The monotone pseudoflow algorithm.

The work done per merger is $O(n)$ since an admissible path is of length $O(n)$. Thus, total work done in mergers including pushes, updating the excesses, and maintaining the arcs in the current forest, is $O(n^2m)$.

Each arc (u, v) needs to be scanned at most once for each value of $\ell(u)$ to determine if it is admissible since node labels are non-decreasing and $\ell(u) \leq \ell(v) + 1$ by Property 3. Thus, if arc (u, v) were not admissible for some value of $\ell(u)$, it can become admissible only if $\ell(u)$ increases. The number of arc scans is thus $O(nm)$ since there are $O(m)$ residual arcs, and each arc is examined $O(n)$ times.

The work done in relabels is $O(n^2)$ since there are $O(n)$ nodes whose labels are bounded by n.

Finally, we need to bound the work done in the depth-first-search for examining nodes within a component. Each time a depth-first-search is executed, either a merger is found or at least one node is relabeled. Thus, the number of times a depth-first-search is executed is $O(nm + n^2)$ which is $O(nm)$. The work done for each depth-first-search is $O(n)$, thus total work done is $O(n^2m)$.

Lemma 11.1. *The complexity of the monotone pseudoflow algorithm is* $O(n^2m)$.

Regarding the complexity of the algorithm, [Hoc08] showed that an enhanced variant of the pseudoflow algorithm is of complexity $O(mn \log n)$. That variant uses dynamic trees data structure and is not implemented here. Recently, however, [HO92] showed that the "highest label" version of the pseudoflow algorithm has complexity $O(mn \log \frac{n^2}{m})$ and $O(n^3)$, with and without the use of dynamic trees, respectively.

References

[CH09] B.G. Chandran and D.S. Hochbaum. A computational study of the pseudoflow and push-relabel algorithms for the maximum flow problem. *Operations Research*, 57(2):358 – 376, 2009.

[GR98] Goldberg, A. V., S. Rao. 1998. Beyond the Flow Decomposition Barrier. *Journal of the ACM* 45(5): 783-797.

[Hoc08] D.S. Hochbaum. The pseudoflow algorithm: A new algorithm for the maximum-flow problem. *Operations Research*, 56(4):992–1009, 2008.

[HO12] D.S. Hochbaum and J.B. Orlin. Simplifications and speedups of the pseudoflow algorithm. *Networks*, 2012. (to appear).

12 The minimum spanning tree problem (MST)

Given a connected graph $G = (V, E)$, with weight w_e for all edge in E, find a subset of edges E_T that contains no cycles and so that the spanning tree $G_T = (V_T, E_T)$ of minimum total weight. (Spanning means that $V_T = V$.)

12.1 IP formulation

The decision variables for the IP formulation of MST are:

$$x_e = \begin{cases} 1 & \text{if edge } e \in E_T \\ 0 & \text{otherwise} \end{cases}$$

The constraints of the IP formulation need to enforce that the edges in E_T form a tree. Recall from early lectures that a tree satisfies the following three conditions: It has $n-1$ edges, it is connected, it is acyclic. Also recall that if any two of these three conditions imply the third one. An IP formulation of MST is:

$$\min \quad \sum_{e \in E} w_e x_e \tag{15a}$$

$$\text{s.t.} \quad \sum_{e \in E} x_e = n - 1 \tag{15b}$$

$$\sum_{e \in (S,S)} x_e \leq |S| - 1 \quad \forall\ S \subseteq V \tag{15c}$$

$$x_e \in \{0, 1\} \quad \forall\ e \in E \tag{15d}$$

where (S, S) denotes the set of edges that have both endpoints in S. Inequality (15c) for S enforces the property that if a subset of edges in (S, S) is connected (induces a connected subgraph), then this subset is acyclic. If (S, S) is disconnected and contains a cycle, then there is another set $S' \subset S$ that violates the respective constraint for S'. Therefore the edges in E_T can't form cycles.

We have the following observations.

1. The constraint matrix of problem (15) does not have a network flow structure, and is not totally unimodular. However Jack Edmonds proved that the LP relaxation has integral extreme points.

2. The formulation contains an exponential number of constraints.
3. Even though the LP relaxation has integral extreme points, this does not imply that we can solve the MST problem in polynomial time. This is because of the exponential size of the formulation. Nevertheless, we can use the ellipsoid method with a *separation algorithm* to solve problem (15):
   Relax the set of constraints given in (15c), and solve the remaining problem. Given the solution to such relaxed problem, find one of the relaxed constraints that is violated (this process is called separation) and add it. Resolve the problem including the additional constraints, and repeat until no constraint is violated.
   It is still not clear that the above algorithm solves in polynomial time MST. However, in light of the equivalence between separation and optimization (one of the central theorems in optimization theory), and since we can separate the inequalities (15c) in polynomial time, it follows that we can solve problem (15) in polynomial time. This can be done using the Ellipsoid algorithm for linear programming.
4. An alternative IP formulation of MST is obtained by imposing the connectivity condition (instead of the acyclicity condition). This condition is enforce by the following set of inequalities, which guarantee that there is at least one edge across each cut:

$$\sum_{e \in (S,\bar{S})} x_e \geq 1 \quad \forall\ S \subset V.$$

   Note that, like the set of constraints (15c), the number of these inequalities is exponential.

12.2 Properties of MST

12.2.1 Cut Optimality Condition

Theorem 12.1. *Let T^* be a spanning tree in $G = (V, E)$. Consider edge $[i, j]$ in T^*. Suppose removing $[i, j]$ from T^* disconnects the tree into S and $\bar{S}$. T^* is a MST iff for all such $[i, j]$, for all $[u, v], u \in S, v \in \bar{S}$, $w_{uv} \geq w_{ij}$.*

Proof. Suppose T^* is an MST and $w_{ij} > w_{kl}$ for some $k \in S, l \in \overline{S}$. Let $T' = T^* \bigcup T(\overline{S}) \bigcup [k, l]$. Then T' is a spanning tree of lower weight than T^*.

Given a spanning tree T^* that satisfies the cut optimality condition. Suppose there exists some MST $T' \neq T^*$, with $[k, l] \in T'$ but not in T^*.

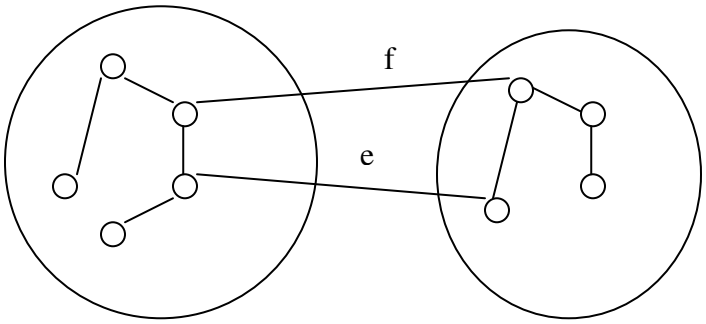


Figure 34: Cut Optimality Condition

We will construct T'' from T^* with one less edge different from T'. Add $[k, l]$ to T^*, creating a cycle. Let $[i, j]$ be the arc with the largest weight in the cycle. Removing this creates a new

spanning tree. But, by the cut optimality condition, $w_{ij} \leq w_{kl}$, but by the optimality of T', $w_{ij} \geq w_{kl}$, so the new tree has the same weight as the old tree. We can repeat this process for all iterations, and find that T^* has the same cost as T'. □

12.2.2 Path Optimality Condition

Theorem 12.2. *T^* is a MST iff for all $[i,j]$ not in T^*, $c_{ij} \geq c_{kl}$ for all $[k,l]$ in the unique path from i to j in T^*.*

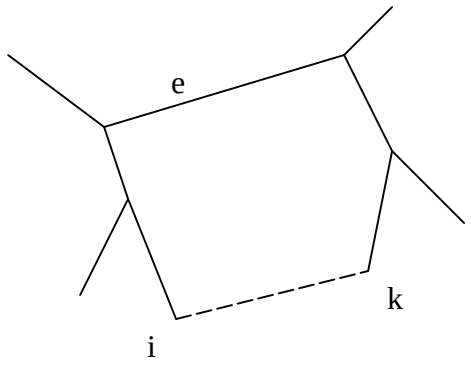


Figure 35: Path optimality condition

Proof. Let T^* be an MST. Suppose $\exists[i,j]$ not in T^* that has a strictly lower cost than an edge in the path from i to j in T^*. Then add $[i,j]$ to T^*, forming a unique cycle, and remove one such edge in the path. The new spanning tree is of strictly lower cost, which is a contradiction.

Let T^* be a spanning tree that satisfies the path optimality condition. Remove edge $[i,j] \in T^*$, disconnecting T^* into S and $\overline{S}$ such that $i \in S, j \in \overline{S}$.

For every edge $[k,l]$ not in T^*, $k \in S, l \in \overline{S}$, the path from k to l must have used $[i,j]$ in T^*. But $c_{ij} \leq c_{kl}$ by path optimality, which is the same statement as cut optimality, so the statements are equivalent. □

12.3 Algorithms of MST

The correctness of the two algorithms for MST given here follows from the following theorem:

Theorem 12.3. *Let $F = \{(U_1, E_1), ...(U_k, E_k)\}$ be a spanning forest (a collection of disjoint trees) in G. Pick edge $[i,j]$ of minimum weight such that $i \in U_1$, $j \in U_1^c$. Then some MST containing $E_1 \bigcup E_2 \bigcup ...E_k$ will contain edge $[i,j]$.*

Proof. Suppose no MST containing F contains edge $[i,j]$. Suppose you are given T^* not containing $[i,j]$. Add $[i,j]$ to E_{T^*}, this will create a cycle, remove the other edge from U_1 to U_1^c that belongs to the created cycle. Since the weight of this removed edge must be greater or equal than that of edge $[i,j]$, this is a contradiction. □

12.3.1 Prim's algorithm

Begin with a spanning forest (in the first iteration let each node being one of the "trees" in this forest). Find the shortest edge $[i, j]$ such that $i \in U_1$ and $j \in U_1^c$), add $[i, j]$ to T_1 and j to U_1. Repeat this process of "growing" U_1 until $U_1 = V$.

begin
$U = \{1\}$; $T = \emptyset$.
while $U \neq V$ **do**
  Pick $[i, j] \in E$ such that $i \in U, j \in U^c$ of minimum weight
  $T \leftarrow T \bigcup [i, j]; P \leftarrow P \bigcup j$
end

The while loop is executed $n - 1$ times, and finding an edge of minimum weight takes $O(m)$ work, so this naive implementation is an $O(nm)$ algorithm. However, note that similarity between this algorithm and Dijkstra's algorithm; in particular note that we can apply the same tricks to improve its running time.

We can make this a faster algorithm by maintaining, in a binary heap, a list of sorted edges.

Each edge is added to the heap when one of its end points enters P, while the other is outside P. It is then removed from the heap when both its end points are in P. Adding an element to a heap takes $O(\log m)$ work, and deleting an element also takes $O(\log m)$ work. Thus, total work done in maintaining the heap is $O(m \log m)$, which is $O(m \log n)$ since m is $O(n^2)$. Thus, total work done in this binary heap implementation is $O(m \log n)$.

With Fibonacci heaps, this can be improved to $O(m + n \log n)$.

For dense graphs, there is a faster, $O(n^2)$ algorithm. This algorithm maintains an array $Q[i]$ for the shortest distance from P to each node $i \in \bar{P}$, outside of P.

begin
  $P = \{1\}, Q[i] = w_{1i}$
  while $P \neq V$ **do**
    Pick $[i, j] \in E$ such that $i \in P, j \in P^c$ of minimum weight
    $T \leftarrow T \bigcup [i, j], P \leftarrow P \bigcup \{j\}$
    for each $[k, j] : k \notin P$ **do**
      $Q[k] \leftarrow \min\{Q[k], w_{kj}\}$
end

This is an $O(n^2)$ algorithm since the while loop is executed $O(n)$ times, and the for loop is executed $O(n)$ times within each while loop. For dense graphs $O(n^2)$ is $O(m)$ and therefore this is an optimal, linear time, algorithm for MST on dense graphs.

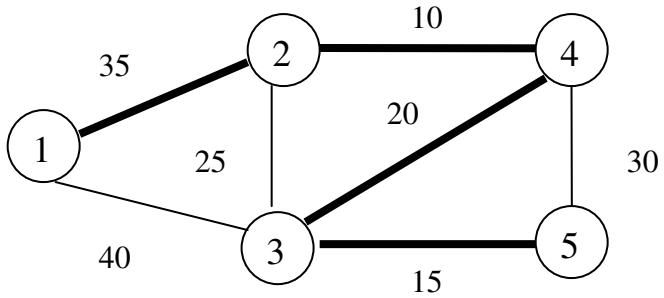


Figure 36: An example of applying Prim's Algorithm

12.3.2 Kruskal's algorithm

Kruskal's algorithm works with a forest and at each iteration it adds an edge that reduces the number of components by one. The algorithm takes as input the ordered list of edges: Sort the edges in non-decreasing order of weight: $w_{e_1} \leq c_{e_2} \leq ...c_{e_m}$.

begin
  $k \leftarrow 1, i \leftarrow 1, E_T \leftarrow \emptyset$
  while $i \leq n$ **do**
    let $e_k = [i, j]$
    if i and j are not connected in (V, E_T) **do**
      $E \leftarrow E \bigcup [i, j]$
      $T \leftarrow T \bigcup \{i, j\}$
    $k \leftarrow k + 1$
end

The algorithm is illustrated in Figure 37.

Complexity: The while loop is executed at most m times, and checking if i and j are in the same component can be done with DFS on (V, E_T), this takes $O(n)$. Therefore this naive implementation has a running time of $O(nm)$.

We will now show that Kruskal's algorithm has a running time of $O(m + n \log n)$ (plus the time to sort the edges) without special data structures:

Each component is maintained as a linked list with a first element label, for each node in the component, as a representative of the component. The size of the component is also stored with the representative "first" of the component.

1. Checking for edge $[i, j]$ whether i and j belong to the same component is done by comparing their labels of component's first. This is done in $O(1)$.
2. If for edge $[i, j]$, i and j belong to the different components, then the edge is added and the two components are merged. In that case the *smaller* component's nodes labels are all updated to assume the label of the representative of the larger component. The size of the component that is the second label of "first" is modified by adding to it the size of the smaller component. The complexity is the size of the smaller component merged.

For each edge $[i, j]$ inspected, checking whether i and j belong to the same component or not takes $O(m)$ operations throughout the algorithm. To evaluate the complexity of merger, note that the "first" label of a node i is updated when i is in the smaller component. After the merger the size of the component that contains i at least doubles. Hence the label update operation on node i can take place at most $O(\log n)$ times. Therefore the total complexity of the labels updates throughout the algorithm is $O(m \log n)$.

Hence, the total complexity of Kruskal's algorithm is $O(m \log n + m + n \log n)$. The complexity of the initial sorting of the edges, $O(m \log n)$, dominates the run time of the algorithm.

12.4 Maximum spanning tree

Note that we never assumed nonnegativity of the edge weights. Therefore we can solve the maximum spanning tree problem with the above algorithms after multiplying the edge weights by -1.

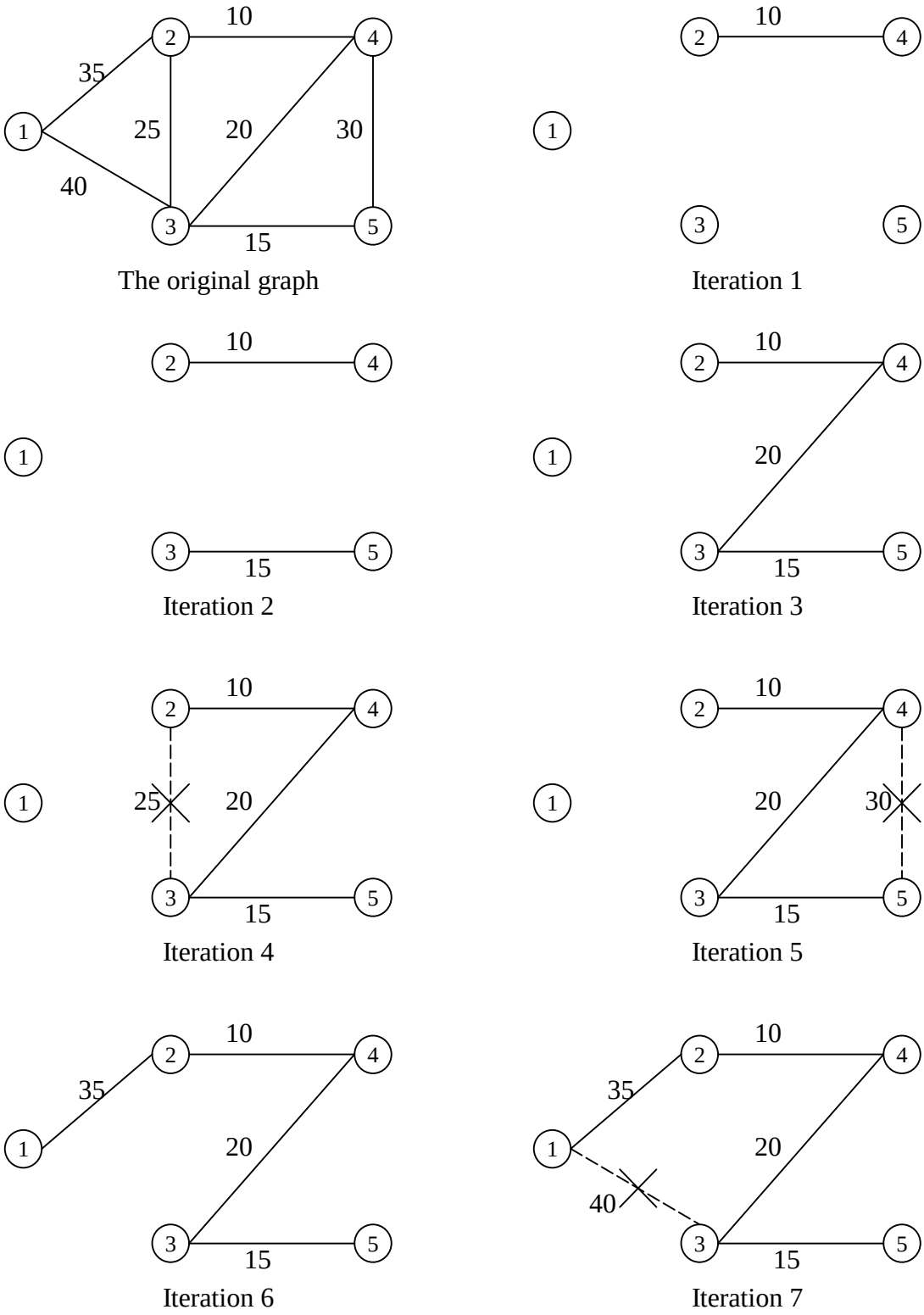


Figure 37: An example of Kruskal's algorithm

13 Complexity classes and *NP*-completeness

In optimization problems, there are two interesting issues: one is *evaluation*, which is to find the optimal value of the objective function (evaluation problems); the other one is *search*, which is to

find the optimal solution (optimization problems).

13.1 Search vs. Decision

Decision Problem - A problem to which there is a yes or no answer.

Example. SAT = {Does there exist an assignment of variables which satisfies the boolean function ϕ; where ϕ is a conjunction of a set of clauses, and each clause is a disjunction of some of the variables and/or their negations?}

Evaluation Problem - A problem to which the answer is the cost of the optimal solution.

Note that an evaluation problem can be solved by solving a auxiliary decision problems of the form "Is there a solution with value less than or equal to M?". Furthermore, using binary search, we only have to solve a polynomial number of auxiliary decision problems.

Optimization Problem - A problem to which the answer is an optimal solution.

To illustrate, consider the Traveling Salesperson Problem (**TSP**). TSP is defined on an undirected graph, $G = (V, E)$, where each edge $(i, j) \in E$ has an associated distance c_{ij}.

TSP_OPT = { Find a tour (a cycle that visits each node exactly once) of total minimum distance. }

TSP_EVAL = { What is the total distance of the tour with total minimum distance in $G = (V, E)$? }

TSP_DEC = { Is there a tour in $G = (V, E)$ with total distance $\leq M$? }

Given an algorithm to solve **TSP_DEC**, we can solve **TSP_EVAL** as follows.

1. Find the upper bound and lower bound for the TSP optimal objective value. Let $C_{min} = \min_{(i,j)\in E} c_{ij}$, and $C_{max} = \max_{(i,j)\in E} c_{ij}$. Since a tour must contain exactly n edges, then an upper bound (lower bound) for the optimal objective value is $n \cdot C_{max}$, $(n \cdot C_{min})$.
2. Find the optimal objective value by binary search in the range $[n \cdot C_{min}, n \cdot C_{max}]$. This binary search is done by calling the algorithm to solve **TSP_DEC** $O(\log_2 n(C_{max} - C_{min})$ times, which is a polynomial number of times.

An important problem for which the distinction between the decision problem and giving a solution to the decision problem – the *search problem* – is significant is primality. It was an open question (until Aug 2002) whether or not there exist polynomial-time algorithms for testing whether or not an integer is a prime number. [2] However, for the corresponding search problem, finding all factors of an integer, no similarly efficient algorithm is known.

Another example: A graph is called k-connected if one has to remove at least k vertices in order to make it disconnected. A theorem says that there exists a partition of the graph into k connected components of sizes $n_1, n_2 \ldots n_k$ such that $\sum_{i=1}^{k} n_k = n$ (the number of vertices), such that each component contains one of the vertices $v_1, v_2 \ldots v_k$. The optimization problem is finding a partition such that $\sum_i |n_i - \bar{n}|$ is minimal, for $\bar{n} = n/k$. No polynomial-time algorithm is known for $k \geq 4$. However, the corresponding recognition problem is trivial, as the answer is always *Yes* once the graph has been verified (in polynomial time) to be k-connected. So is the evaluation problem: the answer is always the sum of the averages rounded up with a correction term, or the sum of the averages rounded down, with a correction term for the residue.

[2] There are reasonably fast superpolynomial-time algorithms, and also Miller-Rabin's algorithm, a randomized algorithm which runs in polynomial time and tells either "I don't know" or "Not prime" with a certain probability. In August 2002 (here is the official release) "Prof. Manindra Agarwal and two of his students, Nitin Saxena and Neeraj Kayal (both BTech from CSE/IITK who have just joined as Ph.D. students), have discovered a polynomial time deterministic algorithm to test if an input number is prime or not. Lots of people over (literally!) centuries have been looking for a polynomial time test for primality, and this result is a major breakthrough, likened by some to the P-time solution to Linear Programming announced in the 70s."

For the rest of the discussion, we shall refer to the decision version of a problem, unless stated otherwise.

13.2 The class *NP*

A very important class of decision problems is called *NP*, which stands for nondeterministic polynomial-time. It is an abstract class, not specifically tied to optimization.

Definition 13.1. *A decision problem is said to be in NP if for all "Yes" instances of it there exists a polynomial-length "certificate" that can be used to verify in polynomial time that the answer is indeed Yes.*

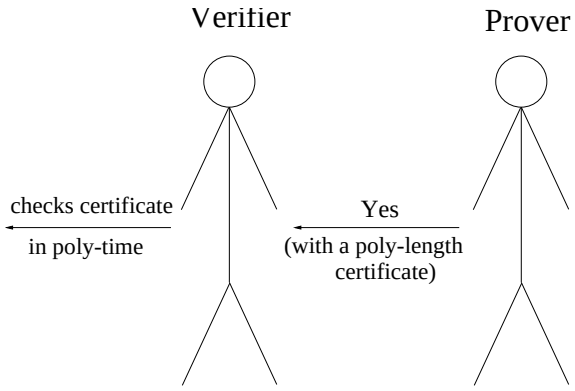


Figure 38: Recognizing Problems in *NP*

Imagine that you are a *verifier* and that you want to be able to confirm that the answer to a given decision problem is indeed "yes". Problems in *NP* have poly-length certificates that, without necessarily indicating how the answer was obtained, allow for this verification in polynomial time (see Figure 38).

To illustrate, consider again the decision version of TSP. That is, we want to know if there exists a tour with total distance $\leq M$. If the answer to our problem is "yes", the prover can provide us with such a tour and we can verify in polynomial time that 1) it is a valid tour and 2) its total distance is $\leq M$. However if the answer to our problem is "no", then (as far as we know) the only way that to verify this would be to check every possible tour (this is certainly not a poly-time computation).

13.2.1 Some Problems in *NP*

Hamiltonian Cycle = {Is there a Hamiltonian circuit in the graph $G = (V, A)$?}
Certificate - the tour.
Clique = {Is there a clique of size $\leq k$ in the graph $G = (V, E)$?}
Certificate - the clique.
Compositeness = {Is N composite (can N be written as $N = ab$, a and b are integers, s.t. $a, b > 1$)? }
Certificate - the factors a and b. (*Note 1:* $\log_2 N$ is the length of the input, length of certificate $\leq 2 \log_2 N$; and we can check in poly-time (multiply). *Note 2:* Giving only one of the factors is also OK.)
Primality = {Is N prime?}

Certificate - not so trivial. In fact, it was only in 1976 that Pratt showed that one exists.

TSP = {Is there a tour in $G = (V, A)$ the total weight of which is $\leq m$?}

Certificate - the tour.

Dominating Set = {Is there is a dominating set $S \subseteq V$ in a graph $G = (V, A)$ such that $|S| \leq k$?}

Certificate - the set. (A vertex is said to dominate itself and its neighbors. A dominating set dominates all vertices)

LP = {Is there a vector $\vec{x}$, $\vec{x} \geq 0$ and $A\vec{x} \leq b$ such that $C\vec{x} \leq k$?}

Certificate - the feasible vector $\vec{x}$. (This vector may be arbitrarily large. However, every basic feasible solution $\vec{x}$ can be shown to be poly-length in terms of the input. Clearly, checkable in polynomial-time.)

IP = {Is there a vector $\vec{x}$, $\vec{x} \geq 0, \vec{x} \in Z$ and $A\vec{x} \leq b$ such that $C\vec{x} \leq k$?}

Certificate - a feasible vector $\vec{x}$. (just like LP, it can be certified in polynomial time; it is also necessary to prove that the certificate is only polynomial in length. see AMO, pg 795.)

k-center = {Given a graph $G = (V, E)$ weighted by $w : V \times V \mapsto R^+$, is there a subset $S \subseteq V$, $|S| = k$, such that $\forall v \in V \setminus S$, $\exists s \in S$ such that $w(v, s) \leq m$?}

Certificate - the subset.

Partition = {Given a list of integers $\{b_1, \ldots, b_n\}$, of sum $\sum_{i=1}^{n} b_i = 2K$, is there a subset $S \subseteq \{1, \ldots, n\}$ such that $\sum_{i \in S} b_i = K$?}

Certificate - the subset.

13.3 *co-NP*

Suppose the answer to the recognition problem is *No*. How would one certify this?

Definition 13.2. *A decision problem is said to be in co-NP if for all "No" instances of it there exists a polynomial-length "certificate" that can be used to verify in polynomial time that the answer is indeed No.*

13.3.1 Some Problems in *co-NP*

Primality = {Is N prime? }

"No" Certificate - the factors.

LP = {Is there a vector $\vec{x}$, $\vec{x} \geq 0$ and $A\vec{x} \leq b$ such that $C\vec{x} \leq k$? }

"No" Certificate - the feasible *dual* vector $\vec{y}$ such that $\vec{y} \leq 0$, $\vec{y}^T A \leq c$ and $\vec{y}^T b > k$ (from Weak Duality Thm.)

13.4 *NP* and *co-NP*

As a verifier, it is clearly nice to have certificates to confirm both "yes" *and* "no" answers; that is, for the problem to be in both *NP* and *co-NP*. From Sections 13.2.1 and 13.3.1, we know that the problems **Prime** and **LP** are in both *NP* and *co-NP*. However, for many problems in *NP*, such as **TSP**, there is no obvious polynomial "no" certificate.

Before we look at these 2 types of problems in detail, we define P , the class of polynomially solvable problems.

Definition 13.3. *Let P be the class of polynomial-time (as a function of the length of the input) solvable problems.*

It is easy to see that $P \subseteq NP \cap co\text{-}NP$ – just find an optimal solution. It is not known, however, whether, $P = NP \cap co\text{-}NP$; the problems that are in the intersection but for which no polynomial-time algorithm is known are usually very pathological. Primality $\in NP \cap co\text{-}NP$; this gave rise to the conjecture that primality testing can be carried out in polynomial time, as indeed it was recently verified to be.

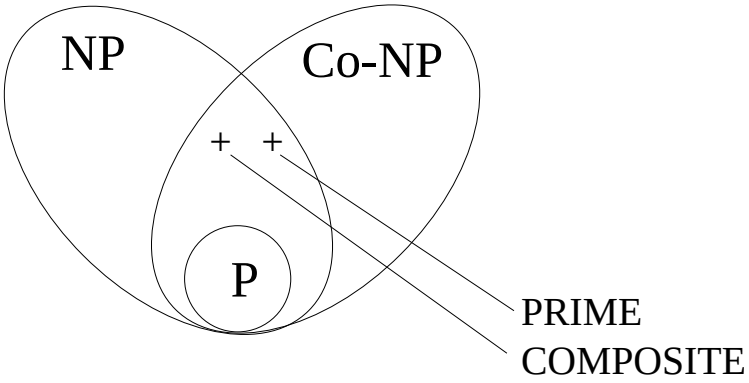


Figure 39: *NP* , *co-NP* , and *P*

Notice that the fact that a problem belongs to $NP \cap co\text{-}NP$ does not automatically suggest a polynomial time algorithm for the problem.

However, being in $NP \cap co\text{-}NP$ is usually considered as strong evidence that there *exists* a polynomial algorithm. Indeed, many problems were known to be in $NP \cap co\text{-}NP$ well before a polynomial algorithm was developed (ex. **LP**, **primality**); this membership is considered to be strong evidence that there was a polynomial algorithm out there, giving incentive for researchers to put their efforts into finding one. Both **primality** and **compositeness** lie in $\in NP \cap co\text{-}NP$; this gave rise to the conjecture that both can be solved in polynomial time.

On the flip-side, *NP*-Complete problems, which we turn to next, are considered to be immune to any guaranteed polynomial algorithm.

13.5 *NP*-completeness and reductions

A major open question in contemporary computer science is whether $P = NP$. It is currently believed that $P \neq NP$.

As long as this conjecture remains unproven, instead of proving that a certain *NP*problem is not in P , we have to be satisfied with a slightly weaker statement: if the problem is in P then $P = NP$, that is, a problem is "at least as hard" as any other *NP*problem. Cook has proven this for a problem called *SAT*; for other problems, this can be proven using *reductions*.

In the rest of this section, we formally define *reducibility* and the complexity class *NP-complete*. Also, we provide some examples.

13.5.1 Reducibility

There are two definitions of reducibility, Karp and Turing; they are known to describe the same class of problems. The following definition uses Karp reducibility.

Definition 13.4. *A problem P_1 is said to reduce in polynomial time to problem P_2 (written as "$P_1 \propto P_2$") if there exists a polynomial-time algorithm A_1 for P_1 that makes calls to a subroutine solving P_2 and each call to a subroutine solving P_2 is counted as a single operation.*

We then say that P_2 is at least as hard as P_1. (Turing reductions allow for only *one* call to the subroutine.)

Important: In all of this course when we talk about reductions, we will always be referring to polynomial time reductions.

Theorem 13.5. *If $P_1 \propto P_2$ and $P_2 \in P$ then $P_1 \in P$*

Proof. Let the algorithm A_1 be the algorithm defined by the $P_1 \propto P_2$ reduction. Let A_1 run in time $O(p_1(|I_1|))$ (again, counting each of the calls to the algorithm for P_2 as one operation), where $p_1()$ is some polynomial and $|I_1|$ is the size of an instance of P_1.

Let algorithm A_2 be the poly-time algorithm for problem P_2, and assume this algorithm runs in $O(p_2(|I_2|))$ time.

The proof relies on the following two observations:

1. The algorithm A_1 can call at most $O(p_1(|I_1|))$ times the algorithm A_2. This is true since each call counts as one operation, and we know that A_1 performs $O(p_1(|I_1|))$ operations.
2. Each time the algorithm A_1 calls the algorithm A_2, it gives it an instance of P_2 of size at most $O(p_1(|I_1|))$. This is true since each bit of the created P_2 instance is either a bit of the instance $|I_1|$, or to create this bit we used at least one operation (and recall that A_1 performs $O(p_1(|I_1|)$ operations).

We conclude that the resulting algorithm for solving P_1 (and now counting all operations) performs at most $O(p_1(|I_1|) + p_1(|I_1|) * p_2((p_1(|I_1|))))$ operations. Since the multiplication and composition of polynomials is still a polynomial, this is a polynomial time algorithm. □

Corollary 13.6. *If $P_1 \propto P_2$ and $P_1 \notin P$ then $P_2 \notin P$*

13.5.2 *NP*-Completeness

A problem $\mathcal{Q}$ is said to be *NP*-hard if $B \propto \mathcal{Q}\ \forall B \in NP$. That is, if all problems in *NP* are polynomially reducible to $\mathcal{Q}$.

A decision problem $\mathcal{Q}$ is said to be *NP*-Complete if:

1. $\mathcal{Q} \in NP$, and
2. $\mathcal{Q}$ is *NP*-Hard.

It follows from Theorem 13.5 that if *any NP*-complete problem were to have a polynomial algorithm, then all problems in *NP* would. Also it follows from Corollary 13.6 that is we prove that *any NP*-complete problem has no polynomial time algorithm, then this would prove that no *NP*-complete problem has a polynomial algorithm.

Note that when a decision problem is *NP*-Complete, it follows that its optimization problem is *NP*-Hard.

Conjecture: P $\neq$ *NP*. So, we do not expect any polynomial algorithm to exist for an *NP*-complete problem.

Now we will show specific examples of *NP*-complete problems.

SAT = {Given a boolean function in conjunctive normal[3] form (CNF), does it have a satisfying assignment of variable? }

$$(X_1 \vee X_3 \vee \bar{X_7}) \wedge (X_{15} \vee \bar{X_1} \vee \bar{X_3}) \wedge (\quad\quad) \ldots \wedge (\quad\quad)$$

Theorem 13.7 (Cook-Levin (1970))**.** *SAT is NP-Complete*

Proof.

SAT is in *NP***:** Given the boolean formula and the assignment to the variables, we can substitute the values of the variables in the formula, and verify in linear time that the assignment indeed satisfies the boolean formula.

$B \propto SAT\ \forall B \in NP$**:** The idea behind this part of the proof is the following: Let X be an instance of a problem $\mathcal{Q}$, where $\mathcal{Q} \in NP$. Then, there exists a SAT instance $F(X, \mathcal{Q})$, whose size is polynomially bounded in the size of X, such that $F(X, \mathcal{Q})$ is satisfiable if and only if X is a "yes" instance of $\mathcal{Q}$.

The proof, in very vague terms, goes as follows. Consider the process of verifying the correctness of a "yes" instance, and consider a Turing machine that formalizes this verification process. Now, one can construct a *SAT* instance (polynomial in the size of the input) mimicking the actions of the Turing machine, such that the *SAT* expression is satisfiable if and only if verification is successful.

□

Karp[4] pointed out the practical importance of Cook's theorem, via the concept of reducibility.

To show that a problem $\mathcal{Q}$ is *NP*-complete, one need only demonstrate two things:

1. $\mathcal{Q} \in NP$, and
2. $\mathcal{Q}$ is *NP*-Hard. However, now to show this we only need to show that $\mathcal{Q}' \propto \mathcal{Q}$, for some *NP*-hard or *NP*-complete problem $\mathcal{Q}'$. (A lot easier than showing that $B \propto \mathcal{Q}\ \forall B \in NP$, right?)

Because all problems in *NP* are polynomially reducible to $\mathcal{Q}'$, if $\mathcal{Q}' \propto \mathcal{Q}$, it follows that $\mathcal{Q}$ is at least as hard as $\mathcal{Q}'$ and that all problems in *NP* are polynomially reducible to $\mathcal{Q}$.

Starting with SAT, Karp produced a series of problems that he showed to be *NP*-complete.[5]

To illustrate how we prove *NP*-Completeness using reductions, and that, apparently very different problems can be reduced to each other, we will prove that the Independent Set problem is *NP*-Complete.

---

[3] In boolean logic, a formula is in conjunctive normal form if it is a conjunction of clauses (i.e. clauses are "linked by and"), and each clause is a disjunction of literals (i.e. literals are "linked by or"; a literal is a variable or the negation of a variable).

[4] Reference: R. M. Karp. *Reducibility Among Combinatorial Problems*, pages 85–103. Complexity of Computer Computations. Plenum Press, New York, 1972. R. E. Miller and J. W. Thatcher (eds.).

[5] For a good discussion on the theory of *NP*-completeness, as well as an extensive summary of many known *NP*-complete problems, see: M.R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-completeness.* Freeman, San Francisco, 1979.

Independent Set (IS) = {Given a graph $G = (V, E)$, does it have a subset of nodes $S \subseteq V$ of size $|S| \geq k$ such that: for every pair of nodes in S there is no edge in E between them?}

Theorem 13.8. *Independent Set is NP-complete.*

Proof.

IS is in *NP***:** Given the graph $G = (V, E)$ and S, we can check in polynomial time that: 1) $|S| \geq k$, and 2) there is no edge between any two nodes in S.

IS is *NP***-Hard:** We prove this by showing that $SAT \propto IS$.

i) Transform input for IS from input for SAT in polynomial time

Suppose you are given an instance of SAT, with k clauses. Form a graph that has a *component* corresponding to each clause. For a given clause, the corresponding component is a complete graph with one vertex for each variable in the clause. Now, connect two nodes in different components if and only if they correspond to a variable and its negation. This is clearly a polynomial time reduction.

ii) Transform output for SAT from output for IS in polynomial time

If the graph has an independent set of size $\geq k$, then our the SAT formula is satisfiable.

iii) Prove the correctness of the above reduction

To prove the correctness we need to show that the resulting graph has an independent set of size at least k **if and only if** the *SAT* instance is satisfiable.

($\rightarrow$) [G has an independent set of size at least $k \rightarrow$ the *SAT* instance is satisfiable]

We create a satisfiable assignment by making true the literals corresponding to the nodes in the independent set; and we make false all other literals. That this assignment satisfies the SAT formula follows from the following observations:

1. The assignment is valid since the independent set may not contain a node corresponding to a variable and a node corresponding to the negation of this variable. (These nodes have an edge between them.)
2. Since 1) the independent set may not contain more than one node of each of the *components* of G, and 2) the value of the independent set is $\geq k$ (it is actually equal to k); then it follows that the independent set must contain exactly one node from each component. Thus every clause is satisfied.

($\leftarrow$) [the *SAT* instance is satisfiable $\rightarrow$ G has an independent set of size at least k]

We create an independent set of size equal to k by including in S one node from each clause. From each clause we include in S exactly one of the nodes that corresponds to a literal that makes this clause true. That we created an independent set of size k follows from the following observations:

1. The constructed set S is of size k since we took exactly one node "from each clause", and the SAT formula has k clauses.
2. The set S is an independent set since 1) we have exactly one node from each component (thus we can't have intra-component edges between any two nodes in S); and 2) the satisfying assignment for SAT is a valid assignment, so it cannot be that both a variable and its negation are true, (thus we can't have

inter-component edges between any two nodes in S).

□

An example of the reduction is illustrated in Figure 40 for the 3-SAT expression $(x_1 \vee x_2 \vee x_3) \wedge (\bar{x_1} \vee \bar{x_2} \vee x_4) \wedge (x_1 \vee \bar{x_3} \vee x_4) \wedge (\bar{x_2} \vee x_3 \vee \bar{x_4})$.

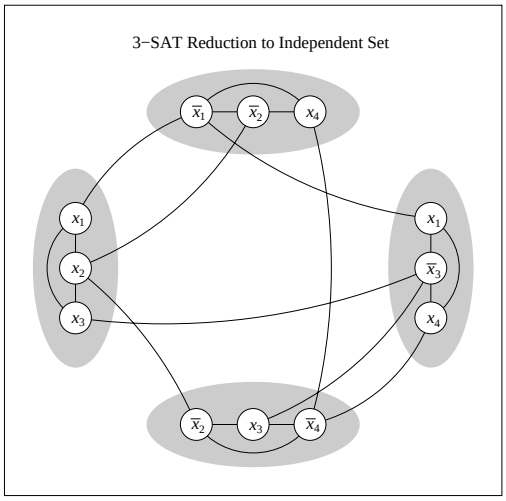


Figure 40: Example of 3-SAT reduction to Independent set

0-1 Knapsack is *NP*-complete.

Proof. First of all, we know that it is in *NP*, as a list of items acts as the certificate and we can verify it in polynomial time.

Now, we show that *Partition* $\propto$ *0-1 Knapsack*. Consider an instance of *Partition*. Construct an instance of *0-1 Knapsack* with $w_i = c_i = b_i$ for $i = 1, 2, \ldots, n$, and $W = K = \frac{1}{2}\sum_{i=1}^{n} b_i$. Then, a solution exists to the *Partition* instance if and only if one exists to the constructed *0-1 Knapsack* instance.

Exercise: Try to find a reduction in the opposite direction. □

k-center is *NP*-complete.

Proof. The *k-center* problem is in *NP* from Section 13.2. The *dominating set* problem is known to be *NP-complete*; therefore, so is *k-center* (from Section 13.5.1). □

About the Knapsack problem:
The Knapsack problem is:
Problem: Given a set of items, each with a weight and value (cost), determine the subset of items with maximum total weight and total cost less than a given budget.

The binary knapsack problem's mathematical programming formulation is as follows.

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} w_j x_j \\ & \sum_{j=1}^{n} v_j x_j \quad \leq \quad B \\ & x_j \quad \in \quad \{0,1\} \end{aligned}$$

The problem can be solved by a longest path procedure on a graph – DAG– where for each $j \in \{1,\ldots,n\}$ and $b \in \{0,1,\ldots,B\}$ we have a node. This algorithm can be viewed alternatively also as a dynamic programming with $f_j(b)$ the value of $\max \sum_{i=1}^{j} u_i x_i$ such that $\sum_{i=1}^{j} v_i x_i \leq b$. This is computed by the recursion:

$$f_j(b) = \max\{f_{j-1}(b); f_{j-1}(b - v_j) + u_j\}.$$

There are obvious boundary conditions,

$$f_1(b) = \begin{cases} 0 & \text{if } b \leq v_1 - 1 \\ u_1 & \text{if } b \geq v_1 \end{cases}$$

The complexity of this algorithm is $O(nB)$. So if the input is represented in *unary* then the input length is a polynomial function of n and b and this run time is considered polynomial. The knapsack problem, as well as any other *NP*-complete problem that has a poly time algorithm for unary input is called *weakly NP-complete*.

14 Approximation algorithms

Approximation algorithms have developed in response to the impossibility of solving a great variety of important optimization problems. Too frequently, when attempting to get a solution for a problem, one is confronted with the fact that the problem is *NP*-hard.

If the optimal solution is unattainable then it is reasonable to sacrifice optimality and settle for a 'good' feasible solution that can be computed efficiently. Of course we would like to sacrifice as little optimality as possible, while gaining as much as possible in efficiency.

Approaches to deal with *NP*-hard problems:

Integer Programming Integer programming tools are forms of implicit enumeration algorithms that combine efficient derivations of lower and upper bounds with a hopeful search for an optimal solution. The amount of time required to solve even a typical moderate-size optimization problem is exorbitant. Instead, the user interrupts the enumeration process either when the current solution is deemed satisfactory, or when the running time has exceeded reasonable limit. The point is that integer programming algorithms provide no *guarantee*. It is impossible to tell if 5 additional minutes of running time would get you a significantly better solution, or if 5 more days of running time would yield no improvement at all.

Heuristics 'Heuristics' is the nomenclature for rules-of-thumb which in itself is a euphemism for simple and invariably polynomial algorithms. Heuristics work quickly and efficiently. The quality of the solution they deliver is another matter altogether: heuristics provide no way to know how close (or far) is the solution found with respect to the optimal solution.

Metaheuristics Metaheuristics are heuristic methods that can be applied to a very general class of computational problems. Some examples are: simulated annealing, genetic algorithms, tabu search, ant colony algorithms, etc. As hueristics, metaheuristics provide no way to know how close (or far) is the solution found with respect to the optimal solution.

Approximation algorithms An approximation algorithm is necessarily polynomial, and is evaluated by the worst case possible relative error over all possible instances of the problem.

Definition 14.1. *A polynomial algorithm, $\mathcal{A}$, is said to be a δ-approximation algorithm for a minimization problem P if for every problem instance I of P with an optimal solution value $OPT(I)$, it delivers a solution of value $\mathcal{A}(I)$ satisfying,*

$$\mathcal{A}(I) \leq \delta OPT(I).$$

Naturally, $\delta > 1$ and the closer it is to 1, the better. Similarly, for maximization problems a δ-approximation algorithm delivers for every instance I a solution that is at least δ times the optimum.

An alternative, and equivalent, definition of approximation algorithms is the following. For a minimization problem, a *δ-approximation algorithm* is a polynomial algorithm, such that for every instance I, the algorithm delivers a value $Alg(I)$ with $Opt(I) \leq Alg(I) \leq \delta \cdot Opt(I)$, where $Opt(I)$ is the optimal solution value. i.e.

$$\text{Sup}_I \left\{ \frac{Alg(I)}{Opt(I)} \right\} \leq \delta$$

An algorithm that is a δ-approximation, guarantees relative error $\leq \delta - 1$:

$$\frac{Alg(I) - Opt(I)}{Opt(I)} \leq \delta - 1 \ .$$

For a maximization problem, $Opt(I) \geq Alg(I) \geq \delta \cdot Opt(I)$, $0 \leq \delta \leq 1$.

14.1 Traveling salesperson problem (TSP)

The Travelling Salesperson Problem (TSP for short) is to visit a given collection of cities (or nodes in a graph) once each and return to the home base while traversing the shortest possible tour. The problem is a well known and studied intractable problem. Here we consider the TSP with triangle inequality, or Δ-inequality satisfied (weights on edges satisfying $\forall_{ijk},\ w_{ij} + w_{jk} \geq w_{ik}$). Even with triangle inequality the problem is still *NP*-hard. However, this problem has a relatively easy 2-approximation algorithm based on the generation of a Minimum Spanning Tree (MST). A minimum spanning tree can be found in a graph polynomially and very efficiently (we will cover this problem in a future lecture). Recall that a graph is Eulerian (or has an Eulerian tour) if and only if every node in the graph has even degree. The Eulerian tour in an Eulerian graph can be constructed quickly - in linear time.

MST-Algorithm:

Step 1: Find a minimum spanning tree (MST) in the graph.

Step 2: "Double" the edges of the spanning tree (to get a graph in which each node has even degree). Construct an Eulerian tour.

Step 3: Construct a valid tour that visits each node exactly once using "short-cuts".

Since the Δ-inequality is satisfied, the shortcut tour is no longer than the Eulerian tour, the length of which is no more than two times the length of the minimum spanning tree (denoted by $|MST|$). Also $|MST| \leq |TSP|$ for $|TSP|$ denoting the length of the optimal tour, because a tour is a spanning tree with one additional edge:

$$\Rightarrow |MST| \leq |TSP| \leq 2|MST| \ .$$

Now,

$$|\text{MST-Algorithm}(I)| \leq 2|MST| \leq 2|TSP| = 2OPT(I)$$

$\Rightarrow$ MST-Algorithm is a 2-approximation algorithm.

A more subtle scheme for approximating TSP with Δ-inequality is due to Christofides (1978). That algorithm uses the fact that some nodes in the spanning tree are of even degree and doubling the edges adjacent to these nodes is superfluous and wasteful. Instead the algorithm due to Christofides works by adding only the bare minimum necessary additional edges to produce an Eulerian graph, thus delivering a better approximation, 3/2-approximation.

Christofides 1.5-approximation Algorithm:

Step 1: Find a minimum spanning tree (MST) in the graph. Let the set of odd degree vertices in MST be V^{odd}.

Step 2: Find the minimum weight perfect matching induced on the set of nodes V^{odd}, M^*. Add the edges of M^* to the edges of MST to create a graph in which each node has even degree. Construct an Eulerian tour.

Step 3: Construct a valid tour that visits each node exactly once using "short-cuts".

We know that the weight of the edges in MST form a lower bound on the optimum. We now show that the weight of the edges in M^* forms a lower bound to $\frac{1}{2}|TSP|$. To see that observe that the optimal tour can be shortcut and restricted to the set of nodes V^{odd} without an increase in cost. Now V^{odd} contains an even number of vertices. An even length tour on V^{odd} can be partitioned to two perfect matchings M_1 and M_2 by taking the alternate set of edges in one, and the remaining alternate set in the other. We thus have

$$|M_1|, \ |M_2| \geq |M^*|.$$

Thus, $|TSP| \geq |M_1| + |M_2| \geq 2|M^*|$ and $|M^*| \leq \frac{1}{2}|TSP|$.

Therefore the algorithm delivered by Christofides' algorithm delivers a tour of length $|MST| + |M^*| \leq \frac{3}{2}|TSP|$.

14.2 Vertex cover problem

The vertex cover problem is defined as follows: Given an undirected graph $G = (V, E)$ we want to find the set $S \subseteq V$ with minimum cardinality such that evert edge in E is adjacent to (at least) one node in S.

In the weighted vertex cover problem each node $i \in V$ has an associated weight $w_i \geq 0$ and we want to minimize the total weight of the set S. Note that, if the graph has negative or zero weights, then we simply include all those nodes in S and remove their neighbors form V. Henceforth we will

refer to the weighted vertex cover problem as the vertex cover problem. The integer programming formulation of the vertex cover problem is given below.

$$\begin{aligned} \min \quad & \sum_{i \in V} w_i x_i \\ \text{s.t.} \quad & x_i + x_j \geq 1 \quad \forall\ (i,j) \in E \\ & x_i \geq 0,\ \textit{integer} \quad \forall\ i \in V \end{aligned}$$

where,

$$x_i \ = \ \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}$$

The vertex cover problem on general graphs is *NP*-hard, however we can solve it in polynomial time on bipartite graphs $G = (V_1 \cup V_2, E)$. This follows since constraint matrix of the vertex cover problem on bipartite graphs is totally unimodular. To see this, observe that we can replace the variables corresponding to the nodes in V_1 by their negative (then $x_i = -1$ if node $i \in S$ and $x_i = 0$ otherwise for all nodes $i \in V_1$). With this change of variables, each row in the constraint matrix has one 1 and one -1.

Moreover, the LP-relaxation of the vertex cover problem on (general) graphs can be solved by solving the vertex cover problem in a related bipartite graph. Specifically, as suggested by Edmonds and Pulleyblank and noted in [NT75], the LP-relaxation can be solved by finding an optimal cover C in a bipartite graph $G_b = (V_{b1} \cup V_{b2}, E_b)$ having vertices $a_j \in V_{b1}$ and $b_j \in V_{b2}$ of weight w_j for each vertex $j \in V$, and two edges $(a_i, b_j), (a_j, b_i)$ for each edge $(i,j) \in E$. Given the optimal cover C on G_b, the optimal solution to the LP-relaxation of our original problem is given by:

$$x_j = \begin{cases} 1 & \text{if} \quad a_j \in C \text{ and } b_j \in C, \\ \frac{1}{2} & \text{if} \quad a_j \in C \text{ and } b_j \notin C, \text{ or} \quad a_j \notin C \text{ and } b_j \notin C, \\ 0 & \text{if} \quad a_j \notin C \text{ and } b_j \notin C. \end{cases}$$

In turn, the problem of solving the vertex cover problem in G_b (or, on any bipartite graph) can be reduced to the minimum cut problem. For this purpose we create a (directed) *st*-graph $G_{st} = (V_{st}, A_{st})$ as follows: (1) $V_{st} = V_b \cup \{s\} \cup \{t\}$, (2) A_{st} contains an infinite-capacity arc (a_i, b_j) for each edge $(a_i, b_j) \in E_b$, (3) A_{st} contains an arc (s, a_i) of capacity w_i for each node $i \in V_{b1}$, and (4) A_{st} contains an arc (b_i, t) of capacity w_i for each node $i \in V_{b2}$. This construction is illustrated in Figure 41.

Given a minimum (S,T) cut we obtain the optimal vertex cover as follows: let $a_i \in C$ if $a_i \in T$ and let $b_j \in C$ if $b_j \in S$.

We now present an alternative method of showing that the LP-relaxation of the vertex cover problem can be reduced to a minimum cut problem. Consider the LP-relaxation of the vertex cover problem (VCR),

$$\begin{aligned} \text{(VCR)} \ \min \quad & \sum_{i \in V} w_i x_i \\ \text{s.t.} \quad & x_i + x_j \geq 1 \quad \forall\ (i,j) \in E \\ & 0 \leq x_i \leq 1 \quad \forall\ i \in V. \end{aligned}$$

Replace each variable x_j by two variables, x_j^+ and x_j^-, and each inequality by two inequalities:

$$\begin{aligned} x_i^+ - x_j^- \ &\geq \ 1 \\ -x_i^- + x_j^+ \ &\geq \ 1\ . \end{aligned}$$

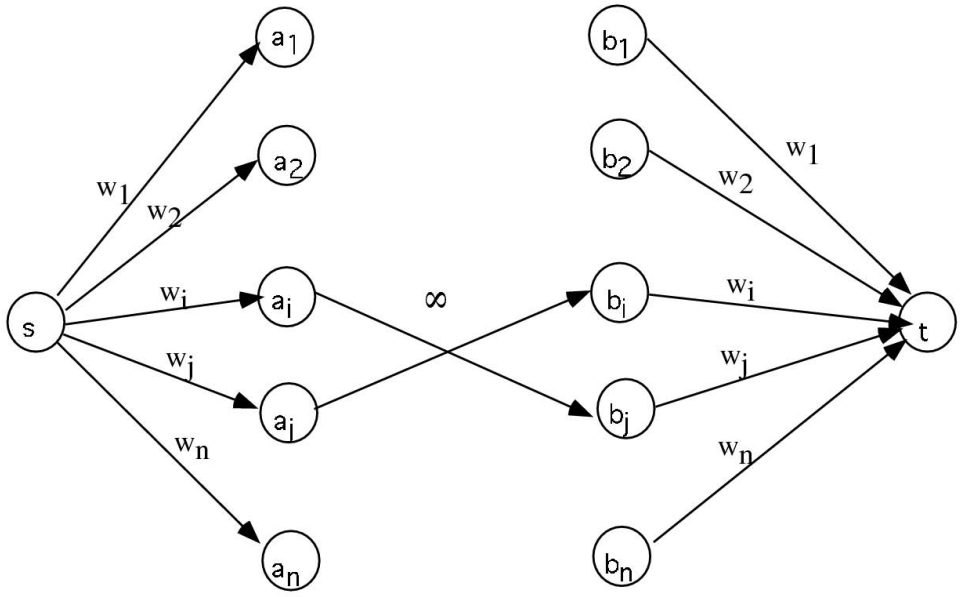


Figure 41: The bipartite network, the minimum cut on which provides the optimal solution to the LP-relaxation of vertex cover

The two inequalities have one 1 and one -1 in each, and thus correspond to a dual of a network flow problem. The upper and lower bounds constraints are transformed to

$$\begin{aligned} 0 \leq x_i^+ &\leq 1 \\ -1 \leq x_i^- &\leq 0 \ . \end{aligned}$$

In the objective function, the variable x_j is substituted by $\frac{1}{2}(x_j^+ - x_j^-)$. All of these result in the following problem:

$$\begin{aligned} \min \quad & \sum_{i \in V} w_i x_i^+ - \sum_{j \in V} w_j x_j^- \\ \text{s.t.} \quad & x_i^+ - x_j^- \geq 1 \quad \forall\ (i,j) \in E \\ & -x_i^- + x_j^+ \geq 1 \quad \forall\ (i,j) \in E \\ & 0 \leq x_i^+ \leq 1 \quad \forall\ i \in V \\ & -1 \leq x_i^- \leq 0 \quad \forall\ i \in V. \end{aligned}$$

The resulting constraint matrix of the new problem is totally unimodular (indeed it is the formulation of the vertex cover problem on the bipartite graph shown in Figure 41). Hence the linear programming (optimal basic) solution is integer, and in particular can be obtained using a minimum cut algorithm. When the original variables are recovered, they are integer multiples of $\frac{1}{2}$. In particular, the original variables can be either $0, \frac{1}{2}$ or 1.

Remark: When the problem is *unweighted*, the network flow that solves the LP relaxation is defined on *simple* networks. These are networks with all arcs of capacity 1, and every node has either one incoming arc, or one outgoing arc. In this case, the arcs in the bipartition of the type (a_i, b_j) can be assigned capacity 1 instead of ∞. For simple networks, Dinic's algorithm for maximum flow works in $O(\sqrt{n}m)$ time - a significant improvement in running time.

14.3 Integer programs with two variables per inequality

Now lets look at the following problem:

$$\begin{aligned} \text{(IP2)}\ \min \quad & \sum_{i\in V} w_i x_i \\ \text{s.t.} \quad & a_{ij}x_i + b_{ij}x_j \geq c_{ij} \quad \forall\ (i,j)\in E \\ & l_i \leq x_i \leq u_i,\ \textit{integer} \quad \forall\ i \in V \end{aligned}$$

This problem is clearly *NP*-hard, since vertex cover is a special case.

There is a 2-approximation algorithm using minimum st-cut on a graph with $\sum_{i=1^n}(u_i - l_i)$ nodes—note that the size of the graph is not polynomial. To see this we introduce the concept of monotone inequality.

A monotone inequality is an inequality of the form $ax - by \geq c$, where $a, b \geq 0$.

For now, lets focus our attention on the integer programming with monotone inequalities:

$$\begin{aligned} \text{(Monotone IP)}\ \min \quad & \sum_{i\in V} w_i x_i \\ \text{s.t.} \quad & a_{ij}x_i - b_{ij}x_j \geq c_{ij} \quad \forall\ (i,j)\in E \\ & l_i \leq x_i \leq u_i,\ \textit{integer} \quad \forall\ i \in V \end{aligned}$$

We first note that (Monotone IP) is *NP*-hard. However, as in the case of the knapsack problem, we can give an algorithm that depends on the "numbers" in the input (recall that knapsack can be solved in $O(nB)$ time, where B is the size of the knapsack)—this algorithms are sometimes referred to as pseudo-polynomial time algorithms. For this purpose we next describe how to solve the monotone integer program as a minimum closure problem. For this purpose we first make a change of variables in our original problem. In particular we write variable x_i as a summation of binary variables $x_i = l_i + \sum_{p=l_i+1}^{u_i} x_i^{(p)}$; and the restriction that $x_i^{(p)} = 1 \implies x_i^{(p-1)} = 1$ for all $p = l_i + 1, \ldots, u_i$. With this change of variables we rewrite (Monotone IP) as follows:

$$\min \quad w_i l_i + \sum_{i\in V} w_i \sum_{p=l_i+1}^{u_i} x_i^{(p)} \tag{16a}$$

$$\text{s.t.} \quad x_i^{(q(p))} \geq x_j^{(p)} \quad \forall\ (i,j)\in E \quad \text{for } p = l_j + 1, \ldots, u_j \tag{16b}$$

$$x_i^{(p)} \leq x_i^{(p-1)} \quad \forall\ i \in V \quad \text{for } p = l_i + 1, \ldots, u_i \tag{16c}$$

$$x_i^{(p)} \in \{0,1\} \qquad \forall\ i \in V \quad \text{for } p = l_i + 1, \ldots, u_i, \tag{16d}$$

where $q(p) \equiv \left\lceil \frac{c_{ij}+b_{ij}p}{a_{ij}} \right\rceil$ and inequality (16b) follows from the the monotone inequalities in the original problem. In particular, since $a, b \geq 0$ then $ax - by \geq c \iff x \geq \frac{c+by}{a}$. Therefore, we have that if $x_j \geq p$ then $x_i \geq \left\lceil \frac{c+bp}{a} \right\rceil = q(p)$. In terms of the newly defined binary variables, this is equivalent to $x_j^{(P)} = 1 \implies x_i^{(q(p))} = 1$.

Problem 16 is a minimum closure problem, which is equivalent to (Monotone IP). It should be noted that the size of the closure problem is not polynomial. However in previous lectures we already showed that the maximum closure problem can be solved by finding the minimum s,t-cut on a related graph.

Now, we can given the 2-approximation algorithm for (non–monotone) integer programming with two variables per inequality.

As we did with vertex cover, we can "monotonize" (IP2). This is done with the following change of variable: $x_i = (x_i^+ - x_i^-)/2$. By performing analogous transformations that we did for vertex cover, we get the following problem:

$$\begin{aligned} \min \quad & \sum_{i \in V} w_i x_i^+ - \sum_{j \in V} w_j x_j^- \\ \text{s.t.} \quad & a_{ij} x_i^+ - b_{ij} x_j^- \leq c_{ij} \quad \forall\ (i,j) \in E \\ & -a_{ij} x_i^- + b_{ij} x_j^+ \leq c_{ij} \quad \forall\ (i,j) \in E \\ & l_i \leq x_i^+ \leq u_i \quad \forall\ i \in V \\ & -u_i \leq x_i^- \leq -l_i \quad \forall\ i \in V. \end{aligned}$$

Note that the constraints are valid since if we add them, then we get the original constraint.

We can solve the problem above in integers (since it is a (monotone IP), we can reduce it to maximum closure, which in turn reduces to the minimum s,t-cut problem). Note that the solution is a lower bound on IP2, because if exists solution then that solution is still valid in IP2 (by the remark in the previous paragraph). In general it is not trivial to round the solution obtained and get an integer solution for our problem. However we can prove that there always exists a way to round that variables and get our desired 2-approximation.

15 Necessary and Sufficient Condition for Feasible Flow in a Network

First the discussion is for zero lower bounds. We assume throughout that the total supply in the graph is equal to the total demand, else there is no feasible solution.

15.1 For a network with zero lower bounds

A cut is a partition of the nodes, $(D, \bar{D})$. We first demonstrate a necessary condition, and then show that it is also sufficient for flow feasibility.

The set D contains some supply nodes and some demand nodes. Let the supply and demand of each node $j \in V$ be denoted by b_j, where b_j is positive if j is a supply node, and negative, if j is a demand node. The cut $(D, \bar{D})$ must have sufficient capacity to accomodate the net supply (which could be negative) that must go out of D.

$$\sum_{j \in D} b_j \leq C(D, \bar{D}) = \sum_{i \in D, j \in \bar{D}} u_{ij}, \qquad \forall D \subset V. \tag{17}$$

This condition must be satisfied for every partition, namely, for every subset of nodes $D \subset V$.

The interesting part is that this condition is also sufficient, and that it can be verified by solving a certain minimum cut problem.

In order to show that the condition is sufficient we construct an auxiliary s,t graph. Let S be the set of all supply nodes, and B be the set of all demand nodes. We append to the graph a source node s with arcs going to all supply nodes with capacity equal to their supplies. We place a sink node t with arcs going from all nodes of B to the sink with capacity equal to the absolute value of the demand at each node. There exists a feasible flow in the original graph only if the minimum s,t cut in this auxiliary graph is at least equal to the total supply $sup = \sum_{b_j > 0} b_j$ (or

total demand, which is equal to it). In other words, since the total supply value is the capacity of some minimum cut – the one adjacent to the source, or the one adjacent to the sink – either one of these cuts must be a minimum cut. In particular, for any partition of V, $(D, \bar{D})$, the cut capacity $C(\{s\} \cup D, \{t\} \cup \bar{D})$ must be at least as big as the total supply:

$$\begin{aligned} sup \leq C(\{s\} \cup D, \{t\} \cup \bar{D}) &= C(\{s\}, D) + C(D, \bar{D}) + C(D, \{t\}) \\ &= \textstyle\sum_{j \in \bar{D}, b_j > 0} b_j + C(D, \bar{D}) + \sum_{j \in D, b_j < 0} -b_j. \end{aligned}$$

Reordering the terms yields,

$$\sum_{j \in D, b_j > 0} b_j + \sum_{j \in D, b_j < 0} b_j \leq C(D, \bar{D}).$$

The left hand side is simply the sum of weights of nodes in D. This necessary condition is therefore identical to the condition stated in (17).

Verification of condition (17).
A restatement of condition (17) is that for each subset of nodes $D \subset V$,

$$\sum_{j \in D} b_j - C(D, \bar{D}) \leq 0.$$

The term on the left is called the s-excess of the set D. In Section **??** it is shown that the maximum s-excess set in a graph can be found by solving a minimum cut problem. The condition is thus satisfied if and only if the maximum s-excess in the graph is of value 0.

15.2 In the presence of positive lower bounds

In this case condition (17) still applies where the cut value $C(D, \bar{D}) = \sum_{i \in D, j \in \bar{D}} u_{ij} - \sum_{u \in \bar{D}, v \in D} \ell_{uv}$

15.3 For a circulation problem

Here all supplies and demands are zero. The condition is thus,
For all $D \subset V$

$$\sum_{i \in D, j \in \bar{D}} u_{ij} - \sum_{u \in \bar{D}, v \in D} \ell_{uv} \geq 0.$$

15.4 For the transportation problem

Here all arc capacities are infinite. The arcs capacities in the s, t graph we construct, as before, are only finite for source adjacent and sink adjacent arcs. Let the transportation problem be given on the bipartite graph $(V_1 \cup V_2, A)$ for V_1 a set of supply nodes, and V_2 a set of demand nodes.

Consider an partition in the graph $(D, \bar{D})$. Since all arc capacities in A are infinite, Condition (17) is trivially satisfied whenever an arc of A is included in the cut. So we restrict our attention to cuts that do not include an arc of A. Such cuts have the set D closed, namely all the successors of the supply nodes in D are also included in D. For closed sets D the cut capacity is 0. The condition is then for each closed set D,

$$\sum_{j \in D b_j > 0} b_j \leq \sum_{j \in D b_j < 0} -b_j.$$

This condition is an extension of Hall's theorem for the existence of perfect matching in bipartite graphs.

(See also pages 194-196 of text)

16 Planar Graphs

Planar graph : A graph is planar if it can be drawn in the plane so no edges intersect. Such a drawing is called planar embedding.

Kuratowsky theorem(1927) *A graph is planar iff it contains no subgraphs homeomorphic to* $K_{3,3}$ *or* K_5.

Face : Faces are regions defined by a planar embedding, where two points in the same face can be reachable from each other along a continuous curve (that does not intersect any edges).

Chromatic number : The minimum number k for which G is k-colorable.

Critical graph : G is called critical if chromatic number of any proper subgraph of G is smaller than that of G.

k-critical graph : A k-critical graph is one that is k-chromatic and critical.

Theorem 16.1. *If G(not necessarily planar) is k-critical, then minimum degree* $\geq k-1$.

Proof. See J.A.Bondy and U.S.R.Murty(1976)

Proposition 16.1. *In a planar graph, there is a node of degree* ≤ 5.

Theorem 16.2 (Heawood's 5-color theorem(1890))**.** *Every planar graph is 5-colorable.*

Proof. By induction on n, the number of nodes. If $n \leq 5$ then the graph is 5-colorable.

For a graph $G(V,E)$ with $n > 5$, assume that $G - \{v\}$ is 5-colorable for any $v \in V$. Now, if for all v, $\deg(v) \leq 4$ or less than 5 colors are used by the neighbors of v, then G is 5-colorable and we are done.

Suppose G is not 5-colorable, that is, G is 6-critical. By above property and theorem, there exists a node, say v, of degree 5. Because G is 6-critical, there exists a 5-vertex coloring, that is a partitioning of into 5 sets each of a different color, of $G-\{v\}$, say $(V_1, V_2, V_3, V_4, V_5)$. Since G is not 5-colorable, v must be adjacent to a vertex of each of the five colors. Therefore we can assume that the neighbors of v in clockwise order about v are v_1, v_2, v_3, v_4 and v_5, where $v_i \in V_i$ for $1 \leq i \leq 5$. Let $G_{1,3}$ be the subgraph of nodes of V_1 and V_3. Now v_1 and v_3 must belong to the same connected component of $G_{1,3}$. For, otherwise, consider the component of $G_{1,3}$ that contains v_1 (see Figure 42). By interchanging the colors 1 and 3 in this component we obtain a new coloring of $G - \{v\}$ in which only 4 colors are assigned to the neighbors of v and G is 5-colorable. Therefore v_1 and v_3 must belong to the same component of $G_{1,3}$.

Let $P_{1,3}$ be a (v_1, v_3)-path in $G_{1,3}$, and let C denote the cycle $vv_1P_{1,3}v_3v$. Since C separates v_2 and v_4, $G_{2,4}$ has at least two separate components, one inside C and another outside C. By interchanging the colors of the component of $G_{2,4}$ in C that contains v_2 we can color v by color 2.

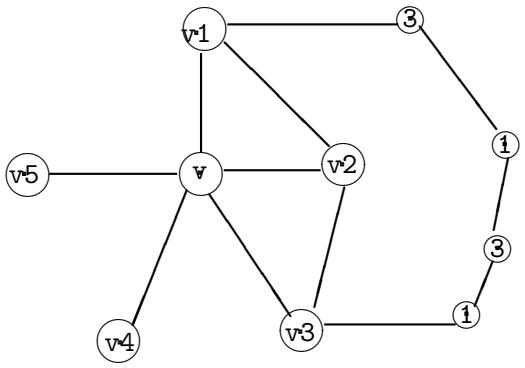


Figure 42: Illustration for proof of 5-Color theorem

16.1 Properties of Planar Graphs

Proposition 16.2 (Euler's Formula)**.** *For a connected planar graph on n nodes, m edges, and f faces, $f = m - n + 2$ (8.6 in text)*

Proof. by induction on the number of faces

If $f = 1$ then the graph is a spanning tree, and it contains $n - 1$ edges. The equation is then satisfied. Assume, inductively, the formula is true for $f \leq k$

Now, we show that the formula is valid for every graph with $k + 1$ faces. Consider a graph G with $k + 1$ faces, n nodes, and $m + 1$ edges. $\exists$ an edge (i, j) separating 2 faces. If we remove (i, j), the resulting graph has k faces, n nodes, and m edges $k = m - n + 2$ (induction hypothesis)

Therefore, if we add (i, j) back into the resulting graph, $k + 1 = m + 1 - n + 2$

Proposition 16.3 (Sparsity)**.** $m < 3n$ *(8.7 in text)*

Proof. This property is proved by contradiction

Suppose $(m \geq 3n)$

Since the network is a simple graph, the boundary of each face contains at least three arcs. If we count arcs on the boundaries of all the faces one by one, there are at least $3f$ arcs. Also, each arc is counted at most twice because it belongs to at most two faces. Therefore, $3f \leq 2m$, or equivalently, $f \leq \frac{2m}{3}$.

By Euler's formula $f = m - n + 2$, so $\frac{2m}{3} \geq m - n + 2$. This, however, implies that $3n \geq m + 6$ which contradicts the supposition that $m \geq 3n$.

Corollary 16.4. *In a planar graph, exist a node with degree ≤ 5*

Proof. Planar graphs are sparse, so $m < 3n$. Let d_i is the degree of node i. $\sum_{i=1}^{n} d_i = 2m < 6n$ Let $d = \sum_{i=1}^{n}$ be the average degree. Sparsity ensures that $d < 6$. Since there must be at least one node j such that $d_j \leq d < 6$. Since d_j is an integer, it follows that $d_j \leq 5$.

Planarity is a hereditary property - in a planar graph, after removing a node the remaining graph is still planar.

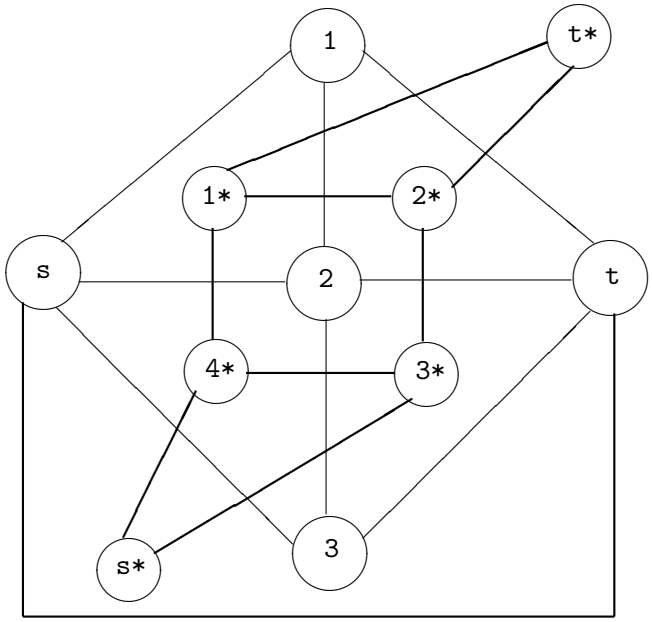


Figure 43: Geometric Dual of a Graph

16.2 Geometric Dual of Planar Graph

The geometric dual of a planar graph is defined by construction:

- 1. Place a node in each face, including the exterior face
- 2. Two nodes are adjacent if the corresponding faces share an edge

The original planar graph is called the primal and the newly constructed graph is called the dual. The dual of a planar graph is still planar.

The followings are some interesting points of primal and dual

- 1. # of nodes in dual = # of faces in primal
- 2. For simple graphs,
  # edges in the primal $\leq$ # edges in the dual
- 3. If no path of length ≥ 2, separates 2 faces,
  then the dual of dual is the primal.

16.3 s-t Planar Flow Network

For an undirected s-t planar flow network s and t are on the exterior face (or on the same face which can then be embedded as an exterior face)

- 1. Add directed arc (t, s) with infinite capacity.

- 2. take the dual of the graph.
- 3. call the two nodes in the two faces created by the new edge e s^* and t^*.

16.4 Min Cut in an Undirected Weighted s-t Planar Flow Network

Weight of edge in dual = capacity of edge in primal

Any path between s^* and t^* corresponds to a cut separating s and t

capacity of cut in primal = cost of path in dual

Shortest path between s^* and t^* corresponds to a minimum cut separating s and t

Since all capacities in primal are nonnegative, all edge costs in dual are nonnegative. Dijkstra's Algorithm can be applied to find the shortest path. The running time for Dijkstra's Algorithm is $O(m + nlogn) = O(3n + nlogn) = O(nlogn)$.

The above algorithm for min cut in planar graph makes no use of the max flow.

16.5 Max Flow in a Undirected Weighted s-t Planar Flow Network

Shown by Hassin in 1981 that max flow can also be derived from shortest path labels.

In the original graph, let U_{ij} be the flow capacity of arc(i,j) X_{ij} be the flow from node i to node j. In an undirected graph, the flow capacity for an edge is $(0, U_{ij})$ for both directions. Therefore $-U_{ij} \leq X_{ij} \leq U_{ij}$

Consider the edge (i,j) in the direction from i to j, let i^* be a dual node in the face to the left of the arc and j^* to the right and define $d(i^*)$ to be the shortest path from s^* to i^* in the dual graph, then set:

$X_{ij} = d(i^*) - d(j^*)$.

Claim 16.5. *$X_{ij} = d(i^*) - d(j^*)$ are feasible flows.*

Proof.
The capacity constraints are satisfied:
Since $d(i^*)$ is the shortest path from s^* to i^* in the dual graph, $d(i^*)$ satisfies the shortest path optimality condition:
$d(i^*) \leq d(j^*) + c_{i^*j^*}$. Since cost of path in dual = capacity of cut in primal

$$d(i^*) \leq d(j^*) + U_{ij}$$
$$d(j^*) \leq d(i^*) + U_{ij}$$
$$-U_{ij} \leq X_{ij} \leq U_{ij}$$

The flow balance constraints are satisfied:
For a node i, let's label it's neighbors 1, 2, ...,p and create the dual graph with nodes 1^*, 2^*, ... p^* (see Figure 44).

$$X_{i1} = d(1^*) - d(2^*)$$
$$X_{i2} = d(2^*) - d(3^*)$$

...

$$X_{ip} = d(p^*) - d(1^*).$$

Sum up all the equations we have: $\sum_{k=1}^{p} X_{ik} = 0$.

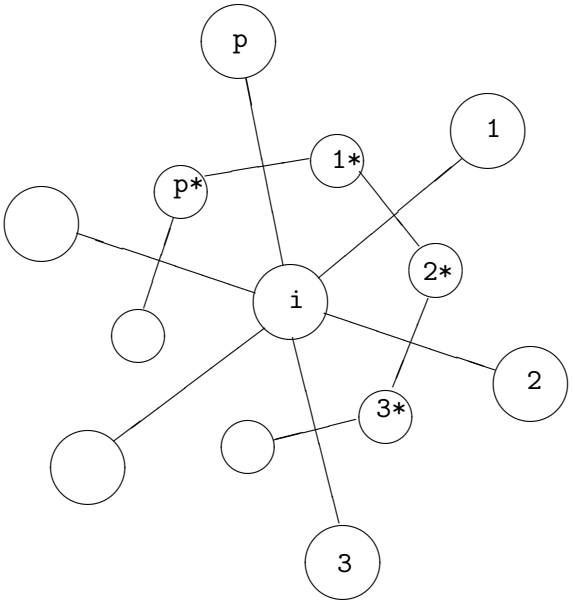


Figure 44: Dual graph for max-flow

For edges on the shortest path in dual, identify the min cut, the flow on the cut edges is saturated, therefore, flow value = cut capacity = max flow

Again we can use Dijkstra's Algorithm to find the shortest path, which is also the max flow. The running time is $O(nlogn)$ as we justified in min cut algorithm.

17 Cut Problems and Algorithms

17.1 Network Connectivity

Network connectivity problems consist of arc connectivity problems and node connectivity problems.

1. The arc connectivity of a connected network is the minimum number of arcs whose removal from the network disconnects it into two or more components. Unweighted arc connectivity has arc weights=1.

2. The node connectivity of a network equals the minimum number of nodes whose deletion from the network disconnects it into two or more components.

Minimum 2-Cut A Minimum 2 cut in a directed network is the min weight arc-connectivity in the network, which is the smallest collection of arcs that separates one non-empty component of the network from the other. That is for each pair of nodes (u, v), find the minimum weight $u - v$ cut and select the minimum of these minimum cuts from among all pairs (u, v).

An Algorithm for Finding a Minimum 2-Cut Find for all pairs $v, u \in V$, the (v-u) - min cut in the network using $O(n^2)$ applications of Max-Flow. This can, however, be done with $O(n)$ applications of Min-Cut. Pick a pair of nodes, $(1, j)$ and solve the min-cut problem for that pair. From among all pairs, take the minimum cut. This requires considering n different pairs of nodes $(1, j)$.

17.2 Matula's Algorithm

Given an unweighted, undirected graph $G = (V, E)$. We introduce the notation $\alpha(G)$ is the connectivity of the graph, which is also the capacity of the minimum 2-cut in the graph. $\alpha(A, B)$ for $A, B \subset V$ disjoint subsets, is the minimum cut separting a and B. Such cut can be found by shrinking A into a source node s, and b into a sink node t and solve the resulting s, t cut problem.

Finally we use the notation $\delta = \min_{v \in V} degree(v)$. δ is obviously an upper bound on the connectivity of the graph, and $nonneighbor(S)$ is the set of nodes that are *not* adjacent to nodes of S.

Let $\alpha(G) = \alpha(S^*, \bar{S}^*)$.

Lemma 17.1 (8.12 in Ahuja, Magnanti, Orlin)**.** *Let $(S^*, \bar{S}^*)$ be a minimum unweighted 2-cut, then either $\alpha(G) = \delta$ or, $\forall S \subseteq S^*$ $nonneighbor(S) \neq \emptyset$*

Proof:

$$\delta|\bar{S}^*| \leq \sum_{v \in \bar{S}^*} degree(v) \leq |\bar{S}^*|(|\bar{S}^*| - 1) + \alpha(G).$$

Subtract δ from both sides,

$$\delta(|\bar{S}^*| - 1) \leq |\bar{S}^*|(|\bar{S}^*| - 1) + \alpha(G) - \delta.$$

Rearranging the terms,

$$1 \leq \delta - \alpha(G) \leq (|\bar{S}^*| - 1)(|\bar{S}^*| - \delta).$$

Therefore both terms on the right must be greater or equal to 1. In particular, $|\bar{S}^*| \geq \delta$. But the number of neighbors of S^* is less than δ, thus, $nonneighbor(S) \neq \emptyset$. □

Matula's Algorithm

$p :=$ min degree node
$\delta \leftarrow degree(p)$
$S^* = \{p\}$, $S = \{p\}$, $\alpha^* \leftarrow \delta$.
While nonneighbor $(S) \neq 0$
 Let $k \in$ nonneighbor (S)
 Find min-cut $(S, k) = (S_k, \bar{S}_k)$ of value $\alpha[S, k]$
 If $\alpha^* > \alpha[S, k]$, then Do
  $\alpha^* \leftarrow \alpha[S, k]$, $S^* \leftarrow S_k$
 Else continue
 $S \leftarrow S \cup \{k\}$

update in-degrees of neighbors of v

Lemma 17.2. *The algorithm is correct. It's complexity is $O(mn)$.*

Proof. First we demonstrate correctness. If $\alpha(G) = \delta$ then done, as we never reduce the value of the cut. Assume then that $\alpha(G) < \delta$. We need to demonstrate that at some iteration $S \subseteq S^*$ and $k \in \bar{S^*}$. This will suffice as that for that iteration $\alpha[S, k] = \alpha^*$.

Initially $S = \{p\} \subseteq S^*$; when the algorithm terminate $S \not\subseteq S^*$, otherwise $nonneighbor(S) \neq \emptyset$, (see lemma). Therefore there must be an iteration i which is the last one in which $S \subseteq S^*$. Consider that iteration and the vertex $k \in noneighbor(S)$ selected. Since $S \cup \{k\} \not\subseteq S^* \Rightarrow k \in \bar{S^*}$. Thus, $\alpha[S, k] = \alpha^*$.

Consider now the complexity of the algorithm: Consider augmentations along shortest augmenting path (such as in Dinic's alg.).

There are two types of augmenting paths:
-type1: its last internal node (before k) $\in neighbor(S)$
-type2: its last internal node (before k) $\in noneighbor(S)$.

The length of type1 path is 2, then the total work for type1 path is $O(n^2)$.

At most there are $O(n)$ type2 path augmentations throughout the alg., in fact consider a path $s, \ldots, l, k$; let l be the node just before k. While k is the sink l can be used just once; as soon as we are done with k, l become a neighbor. Then l can be before the last one only in one type2 path. The complexity of finding an augmenting path is $O(m)$, so the total work for a type2 path is $O(mn)$. Then the total complexity of the algorithm is $O(mn)$.

17.3 Brief Review of Additional Results

Undirected weighted 2-cut: $O(mn + n^2 logn)$ by Nagamochi and Ibaraki, [NI92], [NI92a]. There is no use of the concepts of the max-flow min-cut theorem in their algorithm: It applies preprocessing that removes from the graph all but $O(kn)$ arcs without changing the value of the min 2-cut and then proceed by constructing a collection of $O(k)$ spanning trees.

Directed and undirected weighted 2-cut: $O(n^3)$ or $O(mn \log(n^2/m))$ by Hao and Orlin [HO92]. The idea is to use preflow while maintaining distance labels when values are updated.

References

[HO92] J. Hao and J. B. Orlin. A faster algorithm for finding the minimum cut of a graph. In *Proc. 3rd ACM-SIAM Symp. On Discrete Alg.* 165–174, 1992.

[NI92] H. Nagamochi and T. Ibaraki. Linear time algorithms for finding sparse k-connected spanning subgraph of a k-connected graph. *Algorithmica,* 7 (5/6):583-596, 1992.

[NI92a] H. Nagamochi and T. Ibaraki. Computing edge connectivity in multigraphs and capacitated. *SIAM J. Discrete Math,* 5:54–66, 1992.

18 Algorithms for MCNF

18.1 Network Simplex

Consider the linear programming formulation of the minimum cost flow problem on a network $G = (V, A)$, with x_{ij} the amount of flow on arc $(i, j) \in A$:

$$\begin{aligned}
\text{Minimize} \quad & z = \sum_{i=1}^{n} \sum_{(i,j)\in A} c_{ij} x_{ij} \\
\text{Subject to:} \quad & \sum_{(i,j)\in A} x_{ij} - \sum_{(k,i)\in A} x_{ki} = b_i \quad i \in V \\
& 0 \le x_{ij} \le u_{ij} \qquad\qquad\qquad (i,j) \in A
\end{aligned}$$

It is easily seen that the constraint matrix is not of full rank. In fact, the matrix is of rank n-1. This can be remedied by adding an artificial variable corresponding to an arbitrary node. Another way is just to throw away a constraint, say the one corresponding to node 1. For the uncapacitated version, where the capacities u_{ij} are infinite, the dual is :

$$\begin{aligned}
\text{Maximize} \quad & z = \sum_{j\in V} b_j \pi_j \\
\text{Subject to:} \quad & \pi_i \ - \pi_j \le c_{ij} \quad (i,j) \in A.
\end{aligned}$$

We denote by c_{ij}^{π} the reduced cost $c_{ij} - \pi_i + \pi_j$ which are the slacks in the dual constraints. As proved earlier, the sum of the reduced costs along a cycle is equal to the sum of the original costs.

Simplex method assumes that an initial "basic" feasible flow is given by solving phase I or by solving another network flow problem. By basis, we mean the $n-1$ x_{ij} variables whose columns in the constraint matrix are linearly independent. The non-basic variables are always set to zero. (or in the case of finite upper bounds, nonbasic variables may be set to their respective upper bounds.) We will prove in Section 18.3 that a basic solution corresponds to a spanning tree. The concept of Simplex algorithm will be illustrated in the handout but first we need to define some important terminologies used in the algorithm first.

Given a flow vector x, a basic arc is called *free* iff $0 < x_{ij} < u_{ij}$. An arc (i, j) is called *restricted* iff $x_{ij} = 0$ or $x_{ij} = u_{ij}$. From the concept of basic solution and its corresponding spanning tree we can recognize a characteristic of an optimal solution. An optimal solution is said to be *cycle-free* iff in every cycle at least one arc assumes its lower or upper bound.

Proposition 18.1. *If there exists a finite optimal solution, then there exists an optimal solution which is cycle-free.*

Proof: Prove by contradiction. Suppose we have an optimal solution which is not cycle-free and has the minimum number of free arcs. Then we can identify a cycle such that we can send flow in either clockwise or counter-clockwise direction. By linearity, one direction will not increase the objective value as sum of the costs of arcs along this direction is nonpositive. From our finiteness

assumption, we will make at least one arc achieve its lower or upper bound after sending a certain amount of flow along the legitimate direction. This way, we get another optimal solution having fewer free arcs and the contradiction is found. □

18.2 (T,L,U) structure of an optimal solution

There are three types of arcs in any feasible solution: free arcs, arcs with zero flow, and arcs with the flow at upper bound. We simply state the corollary which stems from the property 18.1 that there exists an optimal solution that is cycle-free.

Corollary 18.2. *There exists an optimal solution that has the structure (T,L,U), where T is a spanning tree containing all free arcs, L is the set of arcs at lower bounds, and U the set of arcs at upper bounds.*

Proof: Since there exists an optimal solution which is cycle-free, then the free arcs form a spanning forest. It is then possible to add restricted arcs to form a spanning tree. □

18.3 Simplex' basic solutions and the corresponding spanning trees

The following is an important observation for applying the simplex method to the minimum cost network flow problem.

Theorem 18.1. *There is a one-to-one correspondence between a basic solution and a spanning tree.*

Proof: A basic solution corresponds to linearly independent columns. Recall that we can add an artificial variable to make the constraint matrix full rank. Clearly, the fictitious column we add is always in a basis for otherwise the original matrix is of full rank. So a basis can be characterized by n nodes and n-1 arcs. If we can show that there are no cycles, then we are done. For the sake of contradiction, suppose there exists a cycle in the graph corresponding to a basis. Select some arc (i,j) in the cycle and orient the cycle in the direction of that arc. Then for each column in the basis associated with an arc in the cycle, assign a coefficient 1 if the arc is in the direction of the orientation of the cycle and -1 otherwise. Applying these coefficients to respective columns in the basis we find that the weighted sum of these columns is the zero vector. Thus the columns of the basis could not have been linearly independent. □

Note that not every basis is feasible, the same as for general linear programming. Furthermore, it can be shown that we can always permute the columns of a basis in a way such that it is in lower-triangular form, which is ideal for Gaussian elimination.

The simplex algorithm works as follows: Given a feasible spanning tree, an entering arc is a nonbasic arc with negative reduced cost, or equivalently, we identify a negative cost cycle by adding it to the spanning tree of basic arcs, all of which have zero reduced costs. The leaving arc, the counterpart of the entering arc, is the bottleneck arc on the residual cycle.

This algorithm corresponds to optimality condition 1 that says that a solution is optimal if the residual graph contains no negative cost cycle (the sum of *reduced* costs along a cycle is equal to the sum of costs along a cycle).

18.4 Optimality Conditions

For a flow x^*, we define a residual network $G(x^*)$ such that every arc (i,j) with capacity u_{ij}, flow x_{ij}, and arc cost c_{ij} in the original network corresponds to two arcs in the residual network: one going from i to j, with cost c_{ij} and flow $u_{ij} - x_{ij}$, the other from j to i with cost $-c_{ij}$ and flow x_{ij}.

There are three equivalent statements of the optimality of flow x^* for the minimum cost flow problems.

1. The following are equivalent to optimality of x^*.
   - x^* is feasible.
   - There is no negative cost cycle in $G(x^*)$.
2. The following are equivalent to optimality of x^*.
   There exists a potentials vector π such that,
   - x^* is feasible.
   - $c_{ij}^{\pi} \geq 0$ for each arc (i,j) in $G(x^*)$.
3. The following are equivalent to optimality of x^*.
   There exists a potentials vector π such that,
   - x^* is feasible,
   - $c_{ij}^{\pi} > 0 \Rightarrow x_{ij}^* = 0$.
   - $c_{ij}^{\pi} = 0 \Rightarrow 0 \leq x_{ij}^* \leq u_{ij}$.
   - $c_{ij}^{\pi} < 0 \Rightarrow x_{ij}^* = u_{ij}$.

Theorem 18.2 (Optimality condition 1)**.** A feasible flow x^* is optimal iff the residual network $G(x^*)$ contains no negative cost cycle.

Proof: $\Leftarrow$ direction.
Let x^0 be an optimal flow and $x^0 \neq x^*$. $(x^0 - x^*)$ is a feasible circulation flow vector. A circulation is decomposable into at most m primitive cycles. The sum of the costs of flows on these cycles is $c(x^0 - x^*) \geq 0$ since all cycles in $G(x^*)$ have non-negative costs. But that means $cx^0 \geq cx^*$. Therefore x^* is a min cost flow.

$\Rightarrow$ direction.
Suppose not, then there is a negative cost cycle in $G(x^*)$. But then we can augment flow along that cycle and get a new feasible flow of lower cost, which is a contradiction. □

Theorem 18.3 (Optimality condition 2)**.** A feasible flow x^* is optimal iff there exist node potential π such that $c_{ij}^{\pi} \geq 0$ for each arc (i,j) in $G(x^*)$.

Proof: $\Leftarrow$ direction.
If $c_{ij}^{\pi} \geq 0$ for each arc (i,j) in $G(x^*)$, then in particular for any cycle C in $G(x^*)$, $\sum_{(i,j)\in C} c_{ij}^{\pi} = \sum_{(i,j)\in C} c_{ij}$. Thus every cycle's cost is nonnegative. From optimality condition 1 it follows that x^* is optimal.

$\Rightarrow$ direction.
x^* is optimal, thus (Opt cond 1) there is no negative cost cycle in $G(x^*)$. Therefore shortest paths distances are well defined. Let $d(i)$ be the shortest paths distances from node 1 (say). Let the node potentials be $\pi_i = -d(i)$. From the validity of distance labels it follows that,

$$c_{ij} + d(i) \geq d(j)$$

Thus, $c^\pi_{ij} = c_{ij} - \pi_i + \pi_j \geq 0$.

□

Theorem 18.4 (Optimality condition 3)**.** A feasible flow x^* is optimal iff there exist node potential π such that,

- $c^\pi_{ij} > 0 \Rightarrow x^*_{ij} = 0$
- $c^\pi_{ij} = 0 \Rightarrow 0 \leq x^*_{ij} \leq u_{ij}$
- $c^\pi_{ij} < 0 \Rightarrow x^*_{ij} = u_{ij}$.

Proof: $\Leftarrow$ direction.
If these conditions are satisfied then in particular for all residual arcs $(i,j) \in G(x^*)$, $c^\pi_{ij} \geq 0$. Hence, optimality condition 2 is satisfied.
$\Rightarrow$ direction.
From Optimality condition 2 $c^\pi_{ij} \geq 0$ for all (i,j) in $G(x^*)$. Suppose that for some (i,j) $c^\pi_{ij} > 0$ but $x^*_{ij} > 0$. But then $(j,i) \in G(x^*)$ and thus $c^\pi_{ji} = -c^\pi_{ij} < 0$ which is a contradiction (to opt con 2). Therefore the first condition on the list is satisfied. The second is satisfied due to feasibility. Suppose now that the third holds, and $c^\pi_{ij} < 0$. So then arc (i,j) is not residual, as for all residual arcs the reduced costs are nonnegative. So then the arc must be saturated and $x^*_{ij} = u_{ij}$ as stated. □

18.5 Implied algorithms – Cycle cancelling

Optimality condition 1 implies an algorithm of negative cycle cancelling: whenever a negative cost cycle is detected in the residual graph it is "cancelled" by augmenting flow along the cycle in the amount of its bottleneck residual capacity.

Finding a negative cycle in the residual graph is done by applying the Bellman-Ford algorithm. If no negative cost cycle is detected then the current flow is optimal. In the simplex method it is easier to detect a negative cost cycle. Since all in-tree arcs have residual cost 0, it is only necessary to find a negative reduced cost arc out of tree. So it seems that simplex is a more efficient form of negative cycle canceling algorithm, isn't it?

The answer is no. The reason is, that while simplex identifies a negative cost cycle more easily than by Bellman Ford algorithm, this cycle is not necessarily in the residual graph. In other words, there can be in the tree and in the basis, arcs that are at their lower or their upper bound. This is precisely the case of *degeneracy*. In that case, although a negative cost cycle was found, the bottleneck capacity along that cycle can be 0. In that case the simplex pivot leads to another solution of equal value.

The complexity of the **cycle canceling algorithm**, is finite. Every iteration reduces the cost of the solution by at least one unit. For maximum arc cost equal to C and maximum capacity value equal to U the number of iterations cannot exceed $2mUC$. The complexity is thus $O(m^2nUC)$, which is not polynomial, but is only exponential in the length of the numbers in the input.

In order to make this complexity polynomial we can apply a *scaling algorithm*. The scaling can be applied to either capacities, or costs, or both.

18.6 Solving maximum flow as MCNF

We know how to represent the maximum flow problem as a minimum cost network flow. We add an arc of cost -1 and infinite capacity between t and s. Since all other costs in the network are 0 the total cost will be minimum when the flow on the arc from t to s is maximum. Maximum flow as minimum cost network flow is described in Figure 45

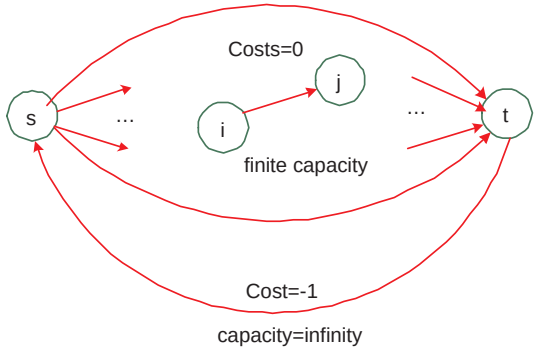


Figure 45: Maximum flow as MCNF problem

If we solve this MCNF problem by cycle canceling algorithm, we observe that the only negative cost arc is (t, s) and therefore each negative cost cycle is a path from s to t augmented with this arc. When we look for negative cost cycle of minimum mean, that is, the ratio of the cost to the number of arcs, it translates in the context of maximum flow to -1 divided by the length of the s, t path in terms of the number of arcs. The minimum mean corresponds to a path of smallest number number of arcs – shortest augmenting path. You can easily verify the other analogies in the following table.

Minimum cost network flow	**maximum flow**
Negative cost cycle in $G(x)$	Augmenting path in $G(x)$
Cycle canceling algorithm	Augmenting path algorithm
Minimum mean cost cycle canceling	Shortest augmenting path
Most negative cost cycle	Augmenting path (all paths have same cost)

Notice that finding a most negative cost cycle in a network is NP-hard as it is at least as hard as finding a hamiltonian path if one exists. (set all costs to 1 and all capacities to 1 and then finding most negative cost cycle is equivalent to finding a longest cycle.) But finding a minimum

mean cost cycle can be done in polynomial time. The mean cost of a cycle is the cost divided by the number of arcs on the cycle.

18.7 The gap between the primal and dual

For the max flow min cut problem the gap between the primal and dual is significant, as we already discussed. We now investigate the gap between primal and dual for MCNF.

We want to show that for an optimal flow $\mathbf{x}^*$ there is a corresponding set of potentials π_i and vice versa.

Flow $\rightarrow$ potentials: Let $\mathbf{x}^*$ be the flow. The residual graph $G(x^*)$ is connected as there must be at least one arc in the residual graph for each $(i,j) \in A$.

Let the length of the arc $(i,j) \in G(\mathbf{x}^*)$ be:

$$d_{ij} = \begin{cases} c_{ij} & \text{if } (i,j) \in A \\ -c_{ji} & \text{if } (j,i) \in A. \end{cases}$$

Now, the shortest path distances from node 1, $d(i)$, are well defined since there are no negative cost cycles in $G(\mathbf{x}^*)$. Let $\pi_i = -d(i)$. Using Bellman-Ford algorithm the complexity of the shortest paths procedure is $O(mn)$.

Potentials $\rightarrow$ flow :
If $c_{ij}^{\pi} > 0$ then set $x_{ij}^* = 0$,
If $c_{ij}^{\pi} < 0$ then set $x_{ij}^* = u_{ij}$.

As for the remaining flows, they can be anywhere between the lower and upper bounds. By assigning the flows guaranteed to be at the upper and lower bounds, the supplies of nodes will be modified. We now use the maximum flow procedure to find a flow that is feasible on these arcs: connect excess nodes to source and deficit nodes to sink and find a flow that saturates these arcs, that is, – maximum. The complexity of this process is that of finding maximum flow, e.g. $O(mn \log \frac{n^2}{m})$.

If the potentials are a *basic* (optimal) solution, then the arcs with $c_{ij}^{\pi} = 0$ form an acyclic graph – a forest. In that case the potentials provide a basis that corresponds to the (T,L,U) structure. In this case the maximum flow needs to be solved on a tree-network. This can be done more efficiently than solving max-flow on a general network: Hochbaums Pseudoflow algorithm, [Hoc08], was shown to solve the maximum-flow problem on s,t-tree networks in linear time, $O(n)$, where s,t-tree networks are formed by joining a forest of capacitated arcs, with nodes s and t adjacent to any subset of the nodes. Notice that such linear gap, if the basis is given, holds also for the maximum flow reconstructed from min-cut solution. Therefore, solving a MCNF problem, and identifying the (T,L,U) basis structure have different complexities (the latter more than the former).

References

[Hoc08] D.S. Hochbaum. The pseudoflow algorithm: A new algorithm for the maximum-flow problem. *Operations Research*, 56(4):992–1009, 2008.